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An Elementary Proof of Riemann's Hypothesis by the Modified Chi-square Function

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

Riemann's hypothesis has always been a challenge in number theory, thus the present article shows a general (i.e. valid for any non-trivial zero) and elementary (i.e. not using the theory of complex functions) proof of it, in which the constant +1/2 arises by itself and automatically. In addition the method gives the values of all the trivial zeroes of Riemann's function. The following steps are used: The modified chi-square function with its parameters Ω, k and $\omega=\omega(k)$, in one of its four forms $(\pm 1\cdot/)X_k^2(\Omega,x/\omega)$, as the interpolating function of the $\{n^\alpha\}$ progressions and of the sums $\{\sum n^\alpha\}$ with $\alpha\in R$ so that $k=2\pm 2\alpha$ for $\alpha<0$ and $\alpha>0$ respectively; the Euler-MacLaurin formula; the shift real vector operator $\mathbf{\Sigma}\equiv(\Sigma_\alpha\ \Sigma_k)\equiv(\Delta\alpha\ \Delta k)\equiv(+1\ (4\alpha-2))$ in the Euclidean 2D space $(\boldsymbol{\alpha}\ \mathbf{k})$ and its extrusion to the imaginary axis i$\mathbf{t}$, leading to the 3D shift complex vector operator $\mathbf{\Sigma}\equiv(\Sigma_\sigma\ \Sigma_k\ \Sigma_{it})\equiv(\Delta\sigma\ \Delta k\ i\Delta t)\equiv(+1\ (4\sigma-2)\ \ it)$ with norm $\Sigma^2=16\sigma^2-16\sigma+5+t^2$. The condition $\mathbf{\Sigma}=\mathbf{0}$ that is $|\mathbf{\Sigma}|=\Sigma=0$ leads to prove RH.

Keywords: Riemann's hypothesis; numeric progressions; modified chi-square function; umbral calculus.

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1 Introduction

Riemann's hypothesis has always been a challenge in number theory and its proof would be a proficient conclusion leading to further results in many fields not only of mathematics but also of physics [1-11]. Thus, mainly in the latest years, the utmost attention has been devoted to debate, deepen and try to solve this problem with admirable findings, though without reaching the final goal [12-21]. In addition claims have been made to its proof [22-24].

In three previous articles by the same author [25-27] the finite sequences of prime numbers $\{P_m\}$ have been examined experimentally from both the statistical and the analytical viewpoint fitting their differential distribution functions and the finite sequences of their parameter $\{\rho_m\}\equiv\{\ln(m_p)/\ln(P_m)\}$ and of their frequencies $\{f_m\}\equiv\{m_p/P_m\}$ by the modified chi-square function $X_k^2(\Omega,x/\omega)$ with its three parameters Ω, k and $\omega=\omega(k)$. Remarkable unexpected results have been found, among which the scale non-invariance of prime numbers, their scaling laws and their correspondence with the truncated progressions $\{n^\alpha\}$ from both the statistical and the analytical viewpoint, just using the methodology of experimental mathematics.

In the present article the problem of Riemann's hypothesis is examined again fitting the progressions $\{n^\alpha\}$ by the modified chi-square function in one of its four forms $(\pm1\cdot/)X_k^2(\Omega,x/\omega)$, according to the α range, just using analytic considerations, thus facing and solving it analytically in a general (that is holding for any non-trivial zero t_m with the index $m\in N$ ranging from 1 up to ∞) and elementary (that is without using the theory of complex functions) way.

The following steps are used for the proof. First of all, the modified chi-square function with its three parameters Ω, k and $\omega=\omega(k)$, described in Ch. 2, in one of its four forms $(\pm1\cdot/)X_k^2(\Omega,x/\omega)$ is used as an interpolating function of the $\{n^\alpha\}$ progressions and of their sums $\{\sum n^\alpha\}$ with $\alpha\in R$, as reported in Ch. 3. Subsequently, the fourth chapter deals with the Euler-MacLaurin summation formula, together with its references to umbral calculus, so that the summation operation $\sum$ is identified as a shift operation and, from the geometric viewpoint, in the real plane $(\boldsymbol{\alpha}\ \mathbf{k})$ the two lines $k = 2 \pm 2\alpha$ for $\alpha < 0$ and $\alpha > 0$ respectively are set up along which all the progressions $\{n^\alpha\}$ and $\{\sum_{n=1\to N}n^\alpha\}\equiv\{N^{\alpha+1}/(\alpha+1)\}$ lay. Finally, in the fifth chapter, the shift operation $\sum$ is associated with the shift real vector operator $\boldsymbol{\Sigma} \equiv (\Sigma_\alpha\ \Sigma_k) \equiv (\Delta\alpha\ \Delta k) \equiv (+1\ (4\alpha–2))$ in the Euclidean 2D real space $(\boldsymbol{\alpha}\ \mathbf{k})$ while the extrusion to the imaginary axis i$\mathbf{t}$ leads to the 3D shift complex vector operator $\boldsymbol{\Sigma} \equiv (\Sigma_\sigma\ \Sigma_k\ \Sigma_{it}) \equiv (\Delta\sigma\ \Delta k\ i\Delta t) \equiv (+1\ (4\sigma–2)\ it)$ with norm $\Sigma^2=16\sigma^2–16\sigma+5+t^2$. The condition $\zeta(s)=\sum n^{\sigma+it}= 0$ i.e. $\boldsymbol{\Sigma}=\mathbf{0}$ that is $|\boldsymbol{\Sigma}|=\Sigma=0$ leads to prove Riemann's hypothesis. Some examples of numerical checks are presented, along the article, just as experimental confirmations and verifications of the adopted methodology. In addition, the induction principle is used all over the work.

2 The Modified Chi-square Function

In the attempt to investigate whether and how the finite progressions $\{n^\alpha\}$ with $n\in N$ and $\alpha\in R$ can be interpolated, in the present report the innovative approach [25-27] is suggested using the modified chi-square function

$$X_k^2(\Omega,x/\omega) = +[\Omega/(2\Gamma_{k/2})]\cdot[x/(2\omega)]^{k/2–1}\cdot e^{–x/(2\omega)} \tag{1}$$

with $k<2.0$ and $k\in R$ as an interpolating function of the progressions $\{n^\alpha\}$ within the open range $\alpha\in(–1, 0)$. The rationale underlying the entire issue is to use this fit function taking advantage of the adjustment of its three parameters k, Ω and ω. In such a way the modified chi-square function can be regarded as an interpolating function of the finite progressions $\{n^\alpha\}$ just like the x^α function and just as the gamma function $\Gamma_x=\Gamma(x)$ is an interpolating function of factorial numbers n! being $n\in N$ and $x\in R^+$.

The modified chi-square function with k degrees of freedom is a new general form of the standard chi-square function [28,29] used also in statistics [30,31] where the two new parameters Ω and ω have been used and

$\Gamma_{k/2}=\Gamma(k/2)$ is the standard gamma function necessary to the normalization. Its characteristics have already been described elsewhere [25-27] and here it is enough to remark that:

for $k < 2$	$\lim_{x\to 0^+} X_k^2[\Omega,x/\omega] = +\infty$	and	$\lim_{x\to\infty} X_k^2[\Omega,x/\omega] = 0^+$
for $k > 2$	$X_k^2[\Omega,0] = 0$	and	$\lim_{x\to\infty} X_k^2[\Omega,x/\omega] = 0^+$

In addition for k=2 the function $X_k^2(\Omega,x/\omega)$ becomes constant. It is very interesting also to remark some features of its three parameters. The coefficient Ω shifts rigidly the function - and the related curve on the 2D real plane (x, X^2) - up or down; the parameter k is responsible for the shape of the function (either a kind of exponential decay for k<2, or a constant function at k=2 or, for k>2, a function with absolute maximum at $x=(k-2)\cdot\omega$ and then decreasing to 0^+ towards infinity; the decay parameter ω stretches or compresses the function and the related curve along the x axis. In addition, it is easy to check that also the other three forms of the modified chi-square function that is $-X_k^2(\Omega,x/\omega)$ $+1/X_k^2(\Omega,x/\omega)$ and $-1/X_k^2(\Omega,x/\omega)$ are useful functions for the present article, the features of which can be easily derived. As a matter of fact, they can interpolate the progressions $\{n^\alpha\}$ in the open ranges $\alpha\in(-2, -1)$ $\alpha\in(0, +1)$ and $\alpha\in(+1, +2)$ respectively, though with different values of Ω and ω.

3 The Interpolation of Numeric Progressions

The examination concerns the finite progressions $\{n^\alpha\}$ with $\alpha\in(-2, -1)$, $\alpha\in(-1, 0)$, $\alpha\in(0, 1)$, $\alpha\in(1, 2)$ and so on, $\alpha\in R$, with domain $n\in N$ and co-domain R^+, showing that any of them can be interpolated analytically by one of the four kinds of the modified chi-square function $(\pm 1\cdot/)X_k^2(\Omega,x/\omega)$ with its own ad hoc parameters Ω, k and ω.

As a matter of fact it is very simple to check analytically that the modified chi-square function $X_k^2(\Omega,x/\omega)$ interpolates the $\{n^\alpha\}$ progressions with $\forall\alpha\in R$ in the same way as the $f(x)=x^\alpha$ function does, what means that both functions are interpolating functions of $\{n^\alpha\}$ for non-integer values of $n\in R^+$. In other words, any $\{n^\alpha\}$ progression can be continued from the N field to the R^+ field. Actually, it is possible to write again (1) as:

$$X_k^2(\Omega,x/\omega) = \{\Omega/[2\Gamma_{k/2}\cdot(2\omega)^{k/2-1}]\}\cdot x^{k/2-1}\cdot e^{-x/(2\omega)} \tag{2}$$

so that, being the value of Ω (free parameter) at one's own choice and being always $x<<\omega$ as experimentally verified too, it is possible to write

$$X_k^2(\Omega,x/\omega) = 1\cdot x^{k/2-1}\cdot e^{-x/(2\omega)} \approx x^{k/2-1} = x^{\pm\alpha} \tag{3}$$

just choosing $\Omega=2\Gamma_{k/2}\cdot(2\omega)^{k/2-1}$ and simply setting $k/2-1=\pm\alpha$. Thus $k=2 \pm 2\alpha$ so that, from the geometric standpoint, two lines are created in the Euclidean real plane ($\boldsymbol{\alpha}$ **k**) of equations

$$k = 2 + 2\alpha \quad \text{for } \alpha<0 \qquad \text{and} \qquad k = 2 - 2\alpha \quad \text{for } \alpha>0 \tag{4}$$

with real domain $\alpha\in(-\infty,+\infty)$ and real co-domain $k < 2$ along which all the progressions $\{n^\alpha\}$ and all their interpolating functions $X_k^2(\Omega,x/\omega)$ lay. Simple numerical checks can be performed, as shown in the four examples of the next Figs. 1 through 4.

The previous Fig. 1 shows the case of $\{n^\alpha\}\equiv\{n^{-1.2}\}$ interpolated by the function $-X_k^2(B,x/y_o)$ with the following parameters: B=1.0E–16 $k=2+2\alpha=2+2\cdot(-1.2)=-0.4$ $\Gamma_{k/2}= -5.821148568627$ y_o=8.33045883E+13 R=1.000000000000=I <C>=<F>=4.17644568E–9 σ_c=9.71356E–9=σ_F LSS=2.08245E–7 X^2_{test}=4.85305E–46 where the calculations are made for $50M=5E7=5\cdot 10^7$ terms of the progression (the plot shows just 50 data-points) and the values I=R=1.000000000000 mean that they are valid up to the 12^{th} decimal digit, i.e. within the precision of the calculations equal to δ=1E–12. The values of the least square sum LSS and of the X^2_{test} are two further markers of the goodness of the fit.

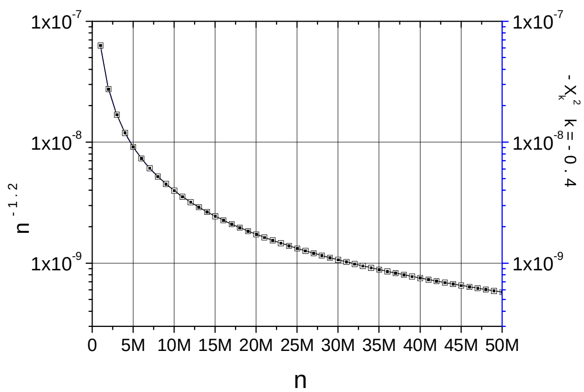

Fig. 1. The progression $\{n^{-1.2}\}$ interpolated by $-X_k^2(B,x/y_o)$ with $k = 2+2\alpha = -0.4$

The Fig. 2 reports the case of $\{n^\alpha\}\equiv\{n^{-0.3}\}$ as fitted by the function $+X_k^2(A,x/x_o)$ for 50 data-points with the following parameters: N=5E7=50M terms A=1.0E–6 $k=2+2\alpha=2+(-0.3)=+1.4$ $\Gamma_{k/2}=1.29805533$ $x_o=6.80082500E+27$ R=1.000000000000=I <C>=<F>=6.764058E–3 $\sigma_c=0.002197808=\sigma_F$ LSS=0.333 $X^2_{test}=2.6494E-26$

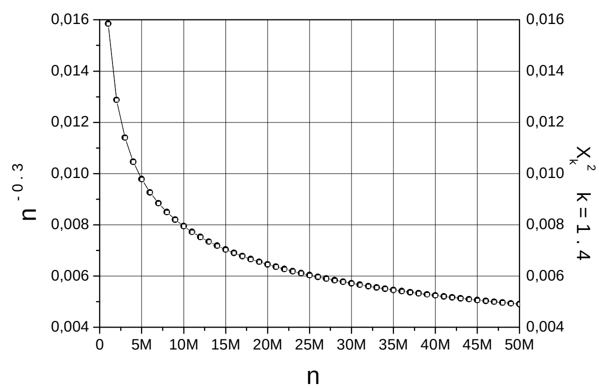

Fig. 2. The progression $\{n^{-0.3}\}$ interpolated by $+X_k^2(A,x/x_o)$ with $k = 2+2\alpha = +1.4$

The Fig. 3 reports the case of $\{n^\alpha\}\equiv\{n^{+0.2}\}$ as interpolated by the function $+1/X_k^2(A,x/x_o)$ with the following parameters: N=5E7=50M terms A=1.0E–6 $k=2-2\alpha=2-2(+0.2)=+1.6$ $\Gamma_{k/2}=1.16422971$ $x_o=3.422264E+31$ R=1.000000000000=I <C>=<F>=29.11700 $\sigma_c=4.621575326$ $\sigma_F=4.621575035$ $X^2_{test}=1.842485E-09$

The Fig. 4 shows the case of $\{n^\alpha\}\equiv\{n^{+1.3}\}$ interpolated by the function $-1/X_k^2(B,x/y_o)$ for 50 data-points with the following features: N=5E7=$5\cdot10^7$ terms B=1.0E–16 $k=2-2\alpha=2-2(+1.3)=-0.6$ $\Gamma_{k/2}=-4.326851108825$ $y_o=5.340564E+12$ R=1.000000000000=I <C>=<F>=4.5378E+09 $\sigma_c=3.091485E9$ $\sigma_F=3.091497E9$

Thus, within the many ranges $\alpha\in(-1, 0)$ $\alpha\in(-2, -1)$ $\alpha\in(-3, -2)$ $\alpha\in(-4, -3)$ and so on, the interpolating functions of all the $\{n^\alpha\}$ progressions are respectively $+X_k^2(A,x/x_o)$ $-X_k^2(B,x/y_o)$ $+X_k^2(C,x/z_o)$$-X_k^2(D,x/w_o)$ in alternation etc. On the contrary, inside the positive ranges $\alpha\in(0, 1)$ $\alpha\in(1, 2)$ $\alpha\in(2, 3)$ $\alpha\in(3, 4)$ and so on the $\{n^\alpha\}$ progressions are interpolated by the functions $+1/X_k^2(A,x/x_o)$ $-1/X_k^2(B,x/y_o)$ $+1/X_k^2(C,x/z_o)$ $-1/X_k^2(D,x/w_o)$ etc.

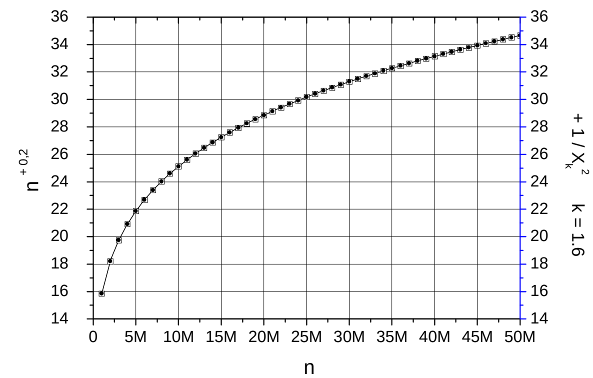

Fig. 3. The progression $\{n^{+0.2}\}$ interpolated by $+1/X_k^2(A,x/x_o)$ with $k = 2–2\alpha = + 1.6$

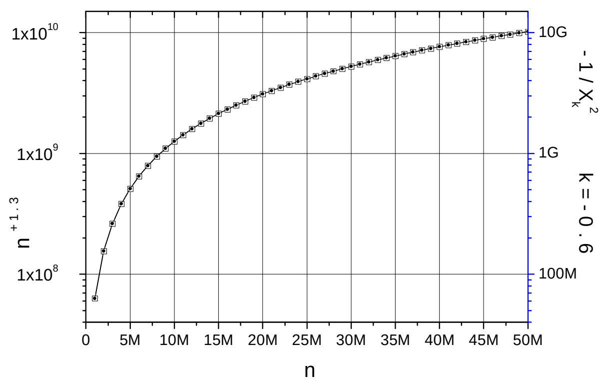

Fig. 4. The progression $\{n^{+1.3}\}$ interpolated by $–1/X_k^2(A,x/x_o)$ with $k = 2–2\alpha = – 0.6$

All that is valid notwithstanding the huge differences among the coefficients A, B, C, D etc. as well as among the decay or growth parameters x_o, y_o, z_o, w_o and so on.

The situation, for just four ranges, is described by the following Table 1 where it is easy to check the symmetry in respect to the value $\alpha=0$ that is $\Omega(\alpha)=\Omega(-\alpha)$ for the coefficients and $\omega(\alpha)=\omega(-\alpha)$ for the decay/growth parameters in the ranges $\alpha\in(\pm j; \pm j+1)$.

In the Table 1 the vector **Σ** will be introduced and discussed later on, while it is important to remark now that all the ranges are open and not closed. As a matter of fact it is easy to check that at the points $(\alpha,k)\equiv(n,2\pm 2n)$ the modified chi-square function is undefined at all: for instance $\{n^\alpha\}\approx +X_k^2(C,x/z_o)$ in the left neighbourhood $(\alpha,k)\equiv(-2^-,-2^-)$ while $\{n^\alpha\}\approx -X_k^2(B,x/y_o)$ in the right neighbourhood $(\alpha,k)\equiv(-2^+,-2^+)$ and the same for any integer value of α. This is also a consequence of the presence of the $1/\Gamma_{k/2}$ function in the modified chi-square function, at least for $\alpha<0$. Moreover, it is interesting to remark that in the left neighbourhood of $(\alpha,k)\equiv(0,2)$ that is at $(0^-,2^-)$ the $+X_k^2(A,x/x_o)=+X_{2^-}^2(A,x/x_o)$ function is the interpolating function of the progression $\{e^{-n}\}$, that is of the function $f(x)=e^{-x}$, while in the right neighbourhood, that is at $(0^+,2^-)$, the $+1/X_k^2(A,x/x_o)=+1/X_{2^+}^2(A,x/x_o)$ function is the interpolating function of the progression $\{e^{+n}\}$, that is of the function $g(x)=e^{+x}$, with $A=2$ and $x_o=1/2$ in both cases.

Table 1. Interpolating functions for the {n^α} progressions within the α and k ranges

α range	α range	k_α=k(α)	$\lvert\Sigma\rvert = \Sigma$	Interpol. funct.
(–2, –1)	(–2, 0)	k=2+2·α	$\sqrt{5}$	$-X_k^2(B,x/y_o)$
(–1, 0)	(0, +2)	k=2+2·α	$\sqrt{[1+(4\alpha-2)^2]}$	$+X_k^2(A,x/x_o)$
(0, +1)	(0, +2)	k=2–2·α	$-\sqrt{5}$	$+1/X_k^2(A,x/x_o)$
(+1, +2)	(–2, 0)	k=2–2·α	$-\sqrt{5}$	$-1/X_k^2(B,x/y_o)$

4 The Euler-Maclaurin Formula

The next step involves the examination of the summation progressions {$\sum n^\alpha$} with the index n running from 1 up to N.

It is well known that, for high enough values of N, the Euler-MacLaurin summation formula applies:

$$\sum n^{\pm\alpha} \sim \int x^{\pm\alpha}dx + \varepsilon = x^{\pm\alpha+1}/({}^{\pm}\alpha+1) + \varepsilon$$

so that it is possible to write

$$\{\textstyle\sum_{n=1\mathrm{thru}N} n^{\pm\alpha}\} \equiv \{N^{\pm\alpha+1}/({}^{\pm}\alpha+1)\} \tag{5}$$

(where the summation runs up to N) for high enough values of N. Actually, it would be better to say that any finite progression {$\sum_{n=1\mathrm{thru}N\to\infty} n^{\pm\alpha}$} is equivalent to $\lim_{N\to\infty}$\{$\sum_{n=1\mathrm{thru}N} n^{\pm\alpha}$\} what is equivalent to $\lim_{N\to\infty}$\{$N^{\pm\alpha+1}/({}^{\pm}\alpha+1)$\}. It is easy to check also experimentally that, whatever the case, the percentage difference between the progression {$\sum n^{\pm\alpha}$} and the next progression {$N^{\pm\alpha+1}/(\pm\alpha+1)$} for an equal number of terms N sufficiently high shows that the limit holds:

$$\lim_{N\to\infty}[(\textstyle\sum_{n=1\mathrm{thru}N} n^{\pm\alpha}) - (N^{\pm\alpha+1})/(\pm\alpha+1)] = 0 \quad \Rightarrow \quad [(\textstyle\sum_{n=1\mathrm{thru}N\to\infty} n^{\pm\alpha}) - (N^{\pm\alpha+1})/(\pm\alpha+1)] \sim 0$$

The situation is fully consistent with the umbral calculus [31-33] where the linear Q-integral operator $L_Q=\int_Q dt$ such that $(L_Q p_n)(t)=\int_Q p_n(t)dt=p_{n+1}(t)/(n+1)$ is introduced, operating on the basic set $(p_n(t))$ of the polynomials in one variable defined over R, together with the shift operator E^a such that $E^a p(t)=p(t+a)$ for $a\in R$ and the Q delta operator that examines $Q(x^n)=a_n x^{n-1}$ in terms of a sequence of real numbers [34].

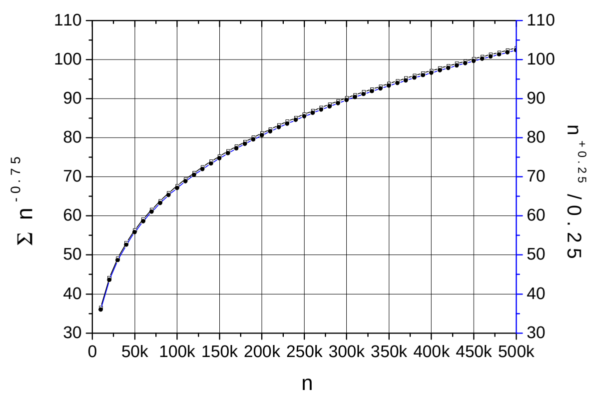

Fig. 5. The progression {$\sum n^{-0.75}$} interpolated by {$N^{+0.25}$ / 0.25} with k'=2–2α'=1.5

Again here too, some numerical checks which are presented in the next two figures can contribute to clarify the situation furthermore.

The Fig. 5 shows the case of the progression $\{\sum n^{\alpha}\}\equiv\{\sum n^{-0.75}\}$ that is $\{\sum n^{-0.75}\}\approx\{N^{0.25}/0.25\}$ with the fit parameter R=0.999999998159 holding for 500K terms of both progressions, for 50 data-points that is 1out-of10K terms.

According to what already told and as experimentally checked, both functions are best fitted by the $1/X_{k'}^{\,2}(A,x/x_o)$ function with $k'=2-2\alpha'=1.5$

Fig. 6 shows the progression $\{\sum n^{\alpha}\}\equiv\{\sum n^{-0.3852}\}$ corresponding to the progression $\{\sum n^{-0.3852}\}\approx\approx\{N^{+0.6148}/0.6148\}$ for 500K terms of both progressions, with the fit parameter R=0.99999999999 and both progressions best fitted by the $1/X_{k'}^{\,2}(A,x/x_o)$ function with $k'= =2-2\alpha'=0.7704$

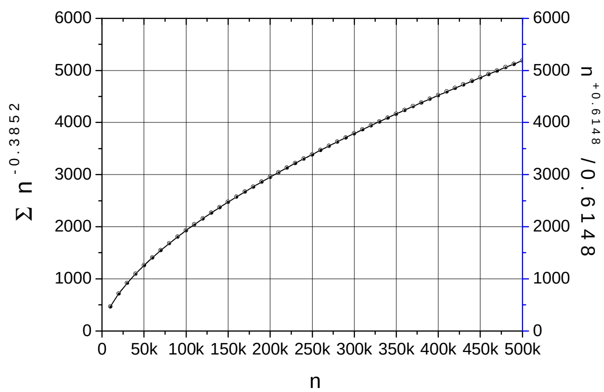

Fig. 6. The progression $\{\sum n^{-0.3852}\}$ interpolated by $\{N^{+0.6148}/0.6148\}$ with $k'=2-2\alpha'=0.7704$

It is interesting to see just an example, from the viewpoint of experimental mathematics, shown in Fig. 7 where the % difference $\Delta\%=[n^{+0.6148}/(0.6148)-\sum n^{-0.3852}]/\sum n^{-0.3852}\cdot 100$ is reported vs. the counter n. Though for just 500K terms (50 data-points in the plot), the asymptotic limit $\lim_{n\to\infty}\Delta\%=0^{-}$ is clear undoubtedly.

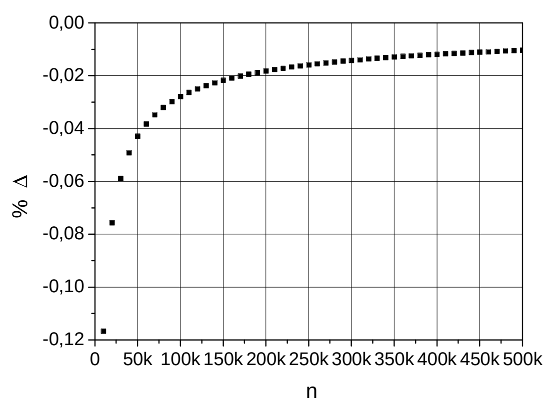

Fig. 7. The % difference between $\{n^{+0.6148} / 0.6148\}$ and $\{\sum n^{-0.3852}\}$

The summation $\sum$ can thus be considered as a shift operation that moves any progression $\{n^{\alpha}\}$ with $\alpha\in(h, h+1)$ up to the next range with $\alpha\in(h+1, h+2)$ that is $\{\sum n^{\alpha}\} \rightarrow \{N^{\alpha+1}/(\alpha+1)\}$ the sum $\sum$ running up to N and the same happens for the progressions $\{n^{-\alpha}\}$ that is at negative values of α. In such a way any $\{\sum n^{-\alpha}\}$ progression corresponds to $\{N^{-\alpha+1}/(-\alpha+1)\}$ with any range $-\alpha\in(h, h+1)$ shifted to the next range that is $-\alpha+1=\alpha'\in(h+1, h+2)$. Owing to the property of scale invariance of the progressions $\{n^{\pm\alpha}\}$ and $\{\sum_{n=1\rightarrow N} n^{\pm\alpha}\}$ the result holds in any case, that is for any value of N, i.e. for an infinite number of terms. In addition, any progression $\{\sum n^{\pm\alpha}\}$ can be associated with the function $\sum_{n=1\rightarrow N} n^{\pm\alpha}$ so that also any function $\zeta(\pm\alpha)=\sum_{n=1\rightarrow N} n^{\pm\alpha}$ corresponds to $N^{\pm\alpha+1}/(\pm\alpha+1)$, that is $\zeta(\pm\alpha)=\sum_{n=1\rightarrow N} n^{\pm\alpha}=N^{\pm\alpha+1}/(\pm\alpha+1)$, or $\int x^{\pm\alpha}d\alpha=x^{\pm\alpha+1}/(\pm\alpha+1)$. In other words, any progression $\{n^{\pm\alpha}\}$ can be interpolated by the function $x^{\pm\alpha}$ and by the function $(\pm 1\cdot/)X_{k'}^{2}(\Omega,x/\omega)$ with $n\in N$, $\alpha\in R$, $x\in R^{+}$, $k'=2\pm 2(\alpha+1)$ and the same holds for all the progressions $\{\zeta(\pm\alpha)\}=\{\sum_{n=1\rightarrow N} n^{\pm\alpha}\}$. Any of them can be interpolated both by $N^{\pm\alpha+1}/(\pm\alpha+1)$ and by the function $(\pm 1\cdot/)X_{k'}^{2}(\Omega,x/\omega)$ with $k'=2\pm 2(\alpha+1)$.

5 Riemann's Zeta Function and Riemann's Hypothesis Proven

An equivalent geometric representation of the situation can be given in the real plane (**α k**) as depicted in the Fig. 8 where the symmetry, in respect to the **k** axis (equation α=0) that is $\Omega(\alpha)=\Omega(-\alpha)$ for the coefficients and $\omega(\alpha)=\omega(-\alpha)$ for the decay or growth parameters, is evident.

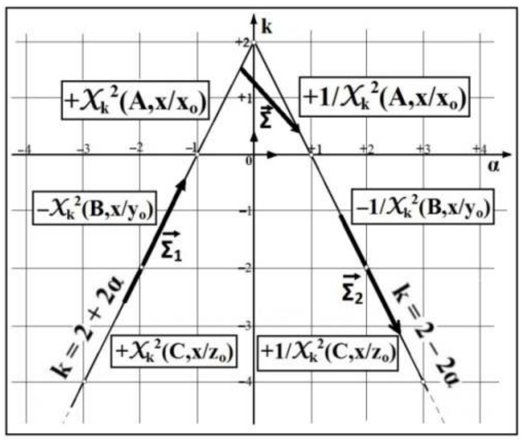

Fig. 8. The lines $k(\alpha) = 2 \pm 2\alpha$ for $\{n^{\alpha}\}$ with the interpolation functions $(\pm 1/\cdot)X_k^2$
Three shift vectors Σ_1 (α<–1) Σ (–1<α<0) and Σ_2 (α>0) are shown as examples

Such a procedure leads to find the values of all the trivial zeroes of Riemann's zeta function (an already known result) and to a proof of Riemann's hypothesis as it is going to be shown.

Hence, as already seen, in the case α<0 writing the function $\sum_{n=1\rightarrow N} n^{-\alpha}$ is the same as writing $(N^{-\alpha+1})/(-\alpha+1)$ and all the progressions $\{\sum n^{-\alpha}\}$, thus the related functions $\zeta(\pm\alpha)=\sum_{n=1\rightarrow N} n^{\pm\alpha}$, lay along the two lines $k = 2 \pm 2\alpha$ on the plane (**α k**). In such a way it is possible to look at the summation operation $\sum$ just as at a shift vector operator $\mathbf{\Sigma_1}$ (as in Fig. 8) that shifts any progression $\{n^{-\alpha}\}$, and thus any function $n^{-\alpha}$, ahead along the line $k=2+2\alpha$ by an amount $\Delta\alpha=+1$ and $\Delta k=+2\Delta\alpha=+2$, that is with norm $|\mathbf{\Sigma_1}|=\Sigma_1=\sqrt{5}$; this happens up to the point $(\alpha, k)\equiv(-1, 0)$ which is shifted by the vector $\mathbf{\Sigma_1}$ up to the point $(\alpha', k')\equiv(0, +2)$. Hence, along this line $k=2+2\alpha$, i.e. $\alpha=k/2-1$, with negative α and within the bounds $\alpha<-1$ and $k<0$ the following relationship holds:

$$\begin{aligned}\sum n^{-\alpha} &= N^{-\alpha+1}/(-\alpha+1) \approx \pm(-\alpha+1)^{-1}\cdot X_{k'}^{2}(\Omega,x/\omega) = \\ &= \pm(-\alpha+1)^{-1}\cdot(\Omega/2\Gamma_{k/2})\cdot(x/2\omega)^{k/2}\cdot e^{x/2\omega} = \\ &= \pm(-\alpha+1)^{-1}\cdot(\Omega/2\Gamma_{1+\alpha})\cdot(x/2\omega)^{\alpha}\cdot e^{(x/2\omega)}\end{aligned}$$

It is clear that in all the cases in which $\sum n^{-\alpha}=\zeta(-\alpha)=0$ that is $\pm X_k^2(\Omega,x/\omega)=0$ the solutions, thus the $X_k^2(\Omega,x/\omega)=\zeta(-\alpha)$ function zeroes, are driven just by the values of $\Gamma_{k/2}=\Gamma_{1+\alpha}$ and consequently the relationship $\sum_{n=1\text{thru}N\to\infty} n^{-\alpha}=\zeta(-\alpha) = (\Omega/2\Gamma_{1+\alpha})\cdot(x/2\omega)^{\alpha}\cdot e^{(x/2\omega)} = 0$ is trivially satisfied at all the points $(\alpha, k)\equiv(\alpha, 2+2\alpha) \equiv (-1,0)\ (-2,-2)\ (-3,-4)\ (-4,-6)\ \ldots\ldots\ (-n,2+2n)$ $n\in N$ being at all these points $1/\Gamma_{k/2} = 1/\Gamma_{1+\alpha} = 1/\Gamma_0 = 1/\Gamma_{-1} = 1/\Gamma_{-2} = 1/\Gamma_{-3} = \ldots\ldots = 0$ These are just the poles of the gamma function $\Gamma_{1+\alpha}=\Gamma_{k/2}$ that is: $\alpha= -1\ -2\ -3\ -4\ldots\ldots -n$ and $k=2+2\alpha= 0\ -2\ -4\ -6\ -8\ \ldots\ (2+2n)$, an already well-known result.

Hence it is very easy to find all the trivial zeroes of Riemann's zeta function $\zeta(-\alpha)=\sum_{n=1\to N} n^{-\alpha}$ in a simple way by the modified chi-square function what confirms not only the usefulness but also the reliability of the algorithm.

The same procedure, as applied to the range $\alpha>0$ with $k=2-2\alpha$ and $k<2$ leads to reject that there are zeroes of the zeta function at positive α values being, in this case,

$$\zeta(\alpha) = \sum n^{\alpha} = N^{\alpha+1}/(\alpha+1) = \pm 1/X_k^{\,2}(\Phi,x/\varphi) = \pm(2\Gamma_{k/2}/\Phi)\cdot(x/2\varphi)^{1-k/2}\cdot e^{x/2\varphi} \neq 0 \qquad \forall\alpha>0$$

It is clear that, in this case, the result is a strict consequence of the fact that the function $\Gamma(x)$ has no poles on the whole R^+ plane being a definite positive function on the whole R^+ plane. The same happens at the point $(\alpha, k)\equiv(0, +2)$ where $\zeta(\alpha)= \zeta(0)=\sum_{n=1\to N} n^0\neq 0$ as already told also about the left and the right neighbourhood of this point.

From the geometric viewpoint, any point $(\pm\alpha, k)$ with any α and $k<2$ laying on the two lines $k= 2 \pm 2\cdot\alpha$ is projected by the generic vector $\mathbf{\Sigma}$ up to the point $(\pm\alpha+1, k+2)$ being: $\Sigma_{1,k}=+2$ $\Sigma_{2,k}=-2$ $\Sigma_k=2\cdot(2\alpha-1)$ $\Sigma_{1,\alpha}=\Sigma_{2,\alpha}=\Sigma_\alpha=+1=\Delta\alpha$ and $|\mathbf{\Sigma}| =\Sigma=\sqrt{(\Sigma_k^{\,2}+ \Sigma_\alpha^{\,2})}=\sqrt{(\Delta\alpha^2+\Delta k^2)}$ the modulus of the vector $\mathbf{\Sigma}$ as in the previous Fig. 8 and the next Fig. 9. In the latter, three typical vectors are highlighted (see also the Table 1): $\mathbf{\Sigma_1}$ for $\alpha<-1$ which has modulus $\Sigma_1=\sqrt{(1^2+2^2)}=\sqrt{5}$, $\mathbf{\Sigma_2}$ for $\alpha>0$ having modulus $\Sigma_2=\sqrt{[1^2+(-2)^2]}=\sqrt{5}$ and $\mathbf{\Sigma}$ for $\alpha\in(-1, 0)$ having varying modulus $\Sigma=\sqrt{[1^2+(4\alpha-2)^2]}=\sqrt{(16\alpha^2-16\alpha+5)}$.

In brief the equation $\sum_{n=1\text{thru}N\to\infty}(n^{-\alpha})=0$ can be formally written using the generic vector operator $\mathbf{\Sigma}$ also as $\mathbf{\Sigma}(n^{-\alpha})=\mathbf{0}$ being $\mathbf{0}$ the null vector, what means $\mathbf{\Sigma}=\mathbf{0}$ i.e. $|\mathbf{\Sigma}|=\Sigma=0$. This artifice is very useful in discussing the next step.

In other words the summation $\sum$ can be regarded as a shift vector operator $\mathbf{\Sigma}$ as depicted in Fig. 9 that shows the 2D situation in the ($\boldsymbol{\alpha}$ $\mathbf{k}$) plane in the range $\alpha\in(-2, +2)$ and $k\in(-2, +2)$.

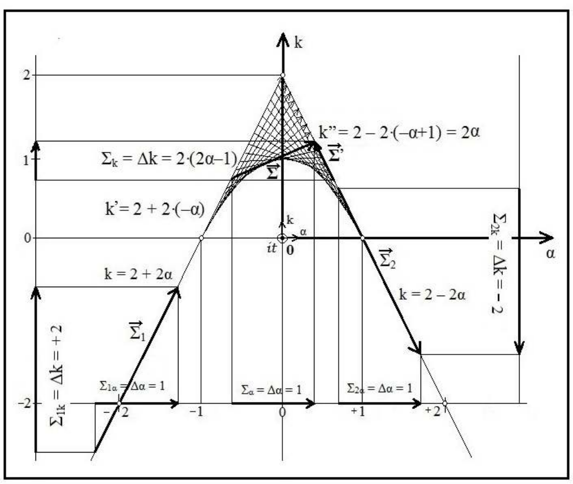

Fig. 9. Examples of the three vectors Σ_1 Σ_2 and Σ along the lines $k=2 \pm 2\alpha$
The envelope of the Σ vectors is the parabola of equation $k = 1 - \alpha^2$

A special attention has to be devoted to the inner range $\alpha\in(-1,0)$ and $k\in(0,2)$ which is the range inside which the $\mathbf{\Sigma}$ vector has a variable norm $|\mathbf{\Sigma}|(\alpha)=\Sigma(\alpha)$; within this range, the line $k=2+2\alpha$ with $\alpha<0$ has been considered as having the equivalent equation $k'=2+2(-\alpha)$ with $\alpha>0$, thus highlighting the negative values of α. Any point $\alpha\in(-1,0)$ and $k\in(0,2)$ of this line with equation $k'=2+2\alpha'=2+2(-\alpha)$ is shifted up to the symmetric line $k''=2-2\alpha''=2-2(\alpha'+\Delta\alpha)=2-2(\alpha'+1)=2-2(-\alpha+1)=2\alpha$ so that the k component of the $\mathbf{\Sigma}$ vector is:

$$\Sigma_k(\alpha) = \Delta k(\alpha) = k''-k' = 2\alpha- [2+2(-\alpha)] = 4\alpha-2$$

Now, considering Euler's real zeta function $\zeta(\alpha)$, the equation $\zeta(\alpha)=\sum_{n=0\text{thru}N\to\infty}n^{-\alpha}=0$ inside this range can be formally written, using the shift vector operator $\mathbf{\Sigma} \equiv (\Sigma_\alpha\ \ \Sigma_k) \equiv (\Delta\alpha\ \Delta k) \equiv (1\ 4\alpha-2)$, also as $\mathbf{\Sigma} n^{-\alpha}=\mathbf{0}$ being $\mathbf{0}$ the **null vector** and $\mathbf{U}$ the **unit vector** $= \mathbf{\Sigma}/|\mathbf{\Sigma}| = \mathbf{\Sigma}/\Sigma$ so that from the geometric viewpoint one can formally write:

$$\mathbf{\Sigma}_{n=0\to\infty}n^{-\alpha} = \mathbf{0} = 0\cdot\mathbf{U} = |\mathbf{\Sigma}|\cdot\mathbf{U}\cdot n^{-\alpha}= \Sigma\cdot\mathbf{U}\cdot n^{-\alpha}$$

getting the equation:

$$\mathbf{\Sigma} = \mathbf{0} \quad\Rightarrow\quad |\mathbf{\Sigma}| = \Sigma = \sqrt{(\Delta\alpha^2+\Delta k^2)} = 0 \quad\Rightarrow\quad \sqrt{(16\alpha^2 - 16\alpha + 5)} = 0$$

with the two complex solutions: $\alpha_{1,2} = +1/2 \pm i/4$

what means, in other words, that the condition $\sum_{n=0\to\infty}(n^{-\alpha})=0$ is satisfied just when the shift vector $\mathbf{\Sigma}$ is horizontal at all i.e. $\Sigma_k=\Delta k=4\alpha-2=0$ that is $\alpha=+1/2$.

Taking into account the extrusion to the third complex axis i**t**, that is the case of the complex exponent $s\in C$ of $\zeta(s)=0$ i.e. $s=\sigma+it$ (replacing the symbol α by σ, thus following the traditional symbolism) one gets for the zeroes of Riemann's zeta function $\zeta(s)= \sum_{n=1\to\infty}(n^{-s})$ the equation

$$\zeta(s) = \zeta(\sigma+it) =\sum_{n=1\to\infty}(n^{-s}) = \sum_{n=1\to\infty}[n^{-(\sigma+it)}] = 0$$

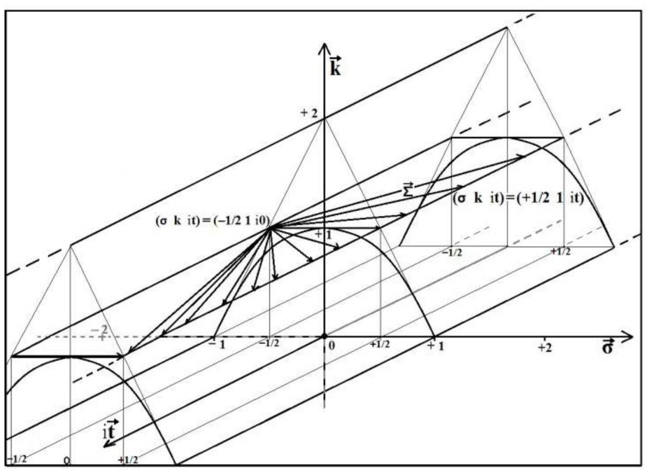

Fig. 10. The extrusion of the real (σ, k) plane to the third imaginary axis it

One should consider that in such a way, from the geometric standpoint, a new axis has been added to the previous two, i.e. the imaginary i**t** axis orthogonal to both the **σ** and the **k** axes that is to the page along which the whole Fig. 9 has been extruded leading to Fig. 10. In such a manner the two previous equations of the lines $k'=2+2\cdot(-\alpha)$ and $k''=2-2\cdot(-\alpha+1)=2\alpha$ now represent the two half-planes having equations $k'=2+2\cdot(-$

σ) and k''=2–2·(–σ+1)=2σ crossing one each other along the line (k=2)∩(σ=0) parallel to the imaginary axis i**t**, thus forming a dihedron (see Fig. 10).

All that implies that the generic 3D shift vector **Σ** has the components $\mathbf{\Sigma} \equiv (\Sigma_\sigma\ \Sigma_k\ \Sigma_t) \equiv (\Delta\sigma\ \Delta k\ i\Delta t) \equiv (1\ 4\sigma{-}2\ it)$ and, owing to the fact that we are in the Euclidean space where the metric is simply the unitary diagonal matrix of components δ_{mn}=Kroneker's symbol ($\delta_{mn}=1$ if m=n and $\delta_{mn}=0$ otherwise), the norm of this new vector operator **Σ** is (using the summation convention):

$$|\mathbf{\Sigma}| = \Sigma = \sqrt{(\Sigma_m\delta_{mn}\Sigma_n)} = \sqrt{[(\Delta\sigma)^2+(\Delta k)^2+t^2]} = \sqrt{[1+(4\sigma-2)^2+t^2]} = \sqrt{(16\sigma^2-16\sigma+5+t^2)}$$

Thus, using the same artifice as above, the equation to be solved now is:

$$\mathbf{\Sigma} n^{-s} = \mathbf{\Sigma}(n^{-(\sigma+it)})=\mathbf{0} \quad \Rightarrow \quad |\mathbf{\Sigma}| = \Sigma = \sqrt{[1+(4\sigma-2)^2+t^2]}=0 \quad \Rightarrow \quad 16\sigma^2-16\sigma+5+t^2=0$$

Solving it leads to the complex solutions:

$$\sigma_{1,2} = +\ 1/2 \pm i/4\cdot\sqrt{(1+t^2)} \ \ldots\ldots\ldots\ldots\ldots \ \forall\ t \tag{6}$$

i.e. taking into account just the positive **t** axis

$$\sigma_m = +\ 1/2 + i/4\cdot\sqrt{(1+t_m^2)} \ \ldots\ldots\ldots\ldots\ldots \ \forall\ t_m \quad \text{i.e.} \quad \forall m \tag{7}$$

thus showing that all the non-trivial zeroes of Riemann's function $\zeta(s)=\zeta(\sigma+it)=\sum_{n=0\to\infty}(n^{-s})=\sum_{n=0\to\infty}(n^{-\sigma-it})$ have real part equal to +1/2 i.e. that they are located on the line Re(s)=+1/2 what is an elementary proof of Riemann's hypothesis as well as of the symmetry of all the complex solutions laying along the line Re(s)=+1/2 in respect to the line $\{s\in C|Im(s)=0\}$ i.e. the real axis **σ** i.e. it=0. This result is general in that valid for any value of t_m that is for any m value (with the index m∈N ranging from 1 up to ∞).

Again, one can look at the situation just from the geometric point of view getting Fig. 11.

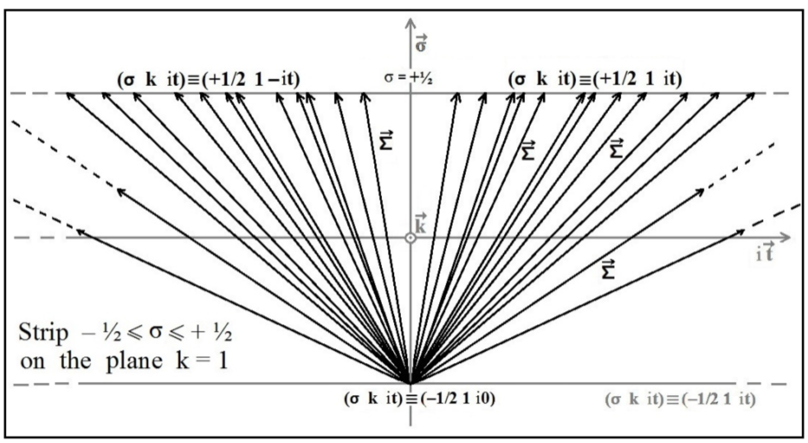

Fig. 11. Riemann's strip on the plane k = + 1 as seen from above
*Some **Σ** vectors laying on the infinitely long strip are shown*

All the **Σ** vectors lay on what can be called Riemann's strip (different from the critical strip) i.e. on the infinitely long strip σ∈[–1/2 +1/2] of the complex plane k=1. The k components Σ_k of all these vectors are null satisfying the condition 4σ–2=0 i.e. σ=+1/2 for ∀t. Remarkably, this outstanding finding has been got in an elementary way that is without using the theory of the functions of complex variable and in a very plain manner.

Furthermore, in the real 2D case the whole geometric procedure is the same as projecting the vector **Σ** onto the line $k''=2-2\cdot(-\alpha+1)=2\alpha$ thus getting the vector **Σ'** such that $\Sigma_k = \Sigma'_k = 4\alpha-2$ which, as shown in Fig. 9, becomes null just at the point

$$(\alpha, k) \equiv (+1/2, 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ i.e. } \alpha = +1/2$$

while in the complex 3D case the whole geometric method is the same as projecting the vector **Σ** onto the plane $k''=2-2\cdot(-\sigma+1)=2\sigma$ thus getting the vector **Σ''** the real component of which, again shown in Fig. 9 as **Σ'**, becomes null just at the point

$$(\sigma, k, it) \equiv (+1/2, 1, it) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ i.e. } \sigma = +1/2 \quad \forall\, t$$

Noticeably, in this elementary proof of Riemann's hypothesis the positive constant +1/2 arises by itself and automatically in a straightforward and natural way.

6 Conclusions

The original and advanced algorithm presented in this article that treats the modified chi-square function, in one of its four forms $\pm(1\cdot/)X_k^2(\Omega,x/\omega)$, as an interpolating function of the progressions $\{n^{\pm\alpha}\}$ - that seemingly might be applied even to other issues of number theory - represents an innovative approach leading to the remarkable result of the proof of Riemann's hypothesis in a general (i.e. holding for any t_m value with the index $m \in N$ from 1 up to ∞) and elementary (i.e. that does not imply the use of the theory of complex functions) way.

In the next future, two main developments, among all the other ones, of the research work will concern both the study of the shift vector operator **Σ** in connection with the Hilbert and Pòlya conjecture and the potential use of the modified chi-square function $X_k^2(\Omega,x/\omega)$ with $k>2$ for the statistical treatment of the normalized spacing of the non-trivial zeroes of Riemann's zeta function, in the frame of random matrices and Gaussian ensembles.

Competing Interests

Author has declared that no competing interests exist.

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