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Sufficient Conditions for CS-recovery

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Original Research Article

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Abstract

In this paper we define the k-restrictly norm constant $r_k(A)$ of a matrix A to be used in compressed sensing and give better error estimations on recovering compressive signals with noise using the matrix $\tilde{A} \equiv \frac{A}{r_k(A)}$. Furthermore, we define the notion of k-restricted invertibility of A, which is equivalent to that $\tilde{A} \equiv A/r_k(A)$ obeys the RIP of order k. And by using the Q. Mo and S. Li idea and T. Cai and A. Zhang idea, we establish the sufficient condition for the restricted isometry constant $\tilde{\delta}_k$ $(k \geq s)$ of $\tilde{A}$ under the assumption that A is k-restrictly invertible. In particular, if $\tilde{\delta}_s < 0.5$ and $\tilde{\delta}_{2s} < 0.828$, then an unknown compressive signal with noise can be recovered.

Keywords: Compressed sensing, Restricted norm constants, Restricted invertible, Restricted isometry constants, Restricted isometry property, Sparse approximation, Sparse signal recovery.

1 Introduction

This paper introduces the theory of compressed sensing(CS). For a signal $\boldsymbol{x} \in \boldsymbol{R}^n$, let $\|\boldsymbol{x}\|_1$ be l_1 norm of $\boldsymbol{x}$ and $\|\boldsymbol{x}\|_2$ be l_2 norm of $\boldsymbol{x}$. Let $\boldsymbol{x}$ be a sparse or nearly sparse vector. Compressed sensing aims to recover high-dimensional signal (for example: images signal, voice signal, code signal...etc.) from only a few samples or linear measurements. Formally, one considers the following model:

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{z}, \tag{1.1}$$

where A is a $m \times n$ matrix($m < n$) and $\boldsymbol{z}$ is an unknown noise term.

Our goal is to reconstruct an unknown signal $\boldsymbol{x}$ based on A and $\boldsymbol{y}$ are given. Then we consider reconstructing $\boldsymbol{x}$ as the solution $\boldsymbol{x}^\star$ to the optimization problem

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1, \quad \text{subject to } \|\boldsymbol{y} - A\boldsymbol{x}\|_2 \leq \varepsilon, \tag{1.2}$$

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where ε is an upper bound on the the size of the noisy contribution.
In fact, a crucial issue is to research good conditions under which the inequality

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq C_0 \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + C_1 \varepsilon, \tag{1.3}$$

for some suitable constants C_0 and C_1, where T_0 is any location of $\{1, 2, \cdots, n\}$ with number $|T_0|$ of elements of T_0 and $\boldsymbol{x}_{T_0}$ is the restriction of $\boldsymbol{x}$ to indices in T_0. One of the most generally known condition for CS theory is the restricted isometry property(RIP) introduced by (1). When we discuss our proposed results, it is an important notion. The RIP needs that the subsets of columns of A for all locations in $\{1, 2, \cdots, n\}$ behave nearly orthonormal system. In detail, a matrix A satisfies the RIP of order s if there exists a constant δ with $0 < \delta < 1$ such that

$$(1 - \delta)\|\boldsymbol{a}\|_2^2 \leq \|A\boldsymbol{a}\|_2^2 \leq (1 + \delta)\|\boldsymbol{a}\|_2^2 \tag{1.4}$$

for all s-sparse vectors $\boldsymbol{a}$. A vector is said to be an s-sparse vector if it has at most s nonzero entries. The minimum δ satisfying the above restrictions is said to be the restricted isometry constant and is denoted by δ_s.
Many researchers has been shown that l_1 optimization can recover an unknown signal in noiseless case and noisy case under various sufficient conditions on δ_s or δ_{2s} when A obeys the RIP. For example, E.J. Candès and T. Tao have proved that if $\delta_{2s} < \sqrt{2} - 1$, then an unknown signal can be recovered (1). Later, S. Foucart and M. Lai have improved the bound to $\delta_{2s} < 0.4531$ (2). Others, $\delta_{2s} < 0.4652$ is used by (3), $\delta_{2s} < 0.4721$ for cases such that s is a multiple of 4 or s is very large by (4), $\delta_{2s} < 0.4734$ for the case such that s is very large by (3) and $\delta_s < 0.307$ by (4). In a resent paper, Q. Mo and S. Li have improved the sufficient condition to $\delta_{2s} < 0.4931$ for general case and $\delta_{2s} < 0.6569$ for the special case such that $n \leq 4s$ (5). T. Cai and A. Zhang have improved the sufficient condition to $\delta_s < 0.333$ for general case. (6). T. Cai and A. Zhang have improved the sufficient condition to δ_k in case of $k \geq \frac{4}{3}s$ in particular $\delta_{2s} < 0.707$. (7). H. Inoue has defined the notion of weak RIP that weakened the notion of RIP and evaluated the solution of CS under the assumption of only the weak RIP. (8). H. Inoue has evaluated the solution of CS without the condition of RIP. (9).
In this paper we define the k-restricted norm constant $r_k(A)$ (simply, r_k) by

$$r_k(A) \equiv \max\{\|A_T\|;\ T \subset \{1, 2, \cdots, n\},\ |T| = k\},$$

where A_T is the $m \times |T|$ matrix composed of these columns for T and $\|\cdot\|$ is operator norm and research good conditions under which the inequality (1.3) holds by considering the following equality (1.5) instead of (1.1):

$$\tilde{\boldsymbol{y}} = \tilde{A}\boldsymbol{x} + \tilde{\boldsymbol{z}} \tag{1.5}$$

where $\tilde{\boldsymbol{y}} \equiv \frac{\boldsymbol{y}}{r_k}$, $\tilde{A} \equiv \frac{A}{r_k}$ and $\tilde{\boldsymbol{z}} \equiv \frac{\boldsymbol{z}}{r_k}$. Since $\{\boldsymbol{a} \in \boldsymbol{R}^n;\ \|\boldsymbol{y} - A\boldsymbol{a}\|_2 \leq \varepsilon\} = \{\boldsymbol{a} \in \boldsymbol{R}^n; \|\tilde{\boldsymbol{y}} - \tilde{A}\boldsymbol{a}\|_2 \leq \frac{\varepsilon}{r_k}\}$, it follows that the solution $\boldsymbol{x}^\star$ to the optimization problem (1.1) is the same as that to (1.5). When A is a matrix with uniformly bounded entries, that is, $|a_{ij}| \leq 1$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$, $r_k(A) \leq \sqrt{ms}$. In particular, if A is the DFT matrix with entries:

$$a_{jt} = e^{-i2\pi jt/n} \quad , 0 \leq j, t \leq n - 1,$$

then $r_k(A) \leq \min\left(\sqrt{mk},\ \sqrt{n}\right)$. We also need the notion of restricted invertibility of A. The matrix A is called k-restrictly invertible if $(A_T^\star A_T)^{\frac{1}{2}}$ is invertible for any subset T of $\{1, 2, \cdots, n\}$ with $|T| = k$.(if and only if A_T is an injection for any subset T of $\{1, 2, \cdots n\}$ with $|T| = k$ if and only if the column vectors $\{\boldsymbol{a}_i;\ i \in T\}$ of A is independent for any T of $\{1, 2, \cdots, n\}$ with $|T| = k$.) It is shown in Lemma 2.1 that A is k-restrictly invertible if and only if $\tilde{A} \equiv \frac{A}{r_k}$ obey the RIP of order k and $r_k(\tilde{A}) \leq 1$. Hence we can make use of results for RIP and in particular, the Candès idea, the Cai et al. idea and the Mo and Li idea. The first main propose is to show that if A is s-restrictly invertible and

$\tilde{\delta}_s < \frac{2}{2+\sqrt{5}} \approx 0.472$, where $\tilde{\delta}_s$ is the restricted isometry constant for $\tilde{A} \equiv \frac{A}{r_s}$, and if A is $2s$-restrictly invertible and the restricted isometry constant $\tilde{\delta}_{2s} < 0.661$, then inequality (1.3) holds. The second is to obtain the better sufficient conditions of $\tilde{\delta}_s < 0.5$ and $\tilde{\delta}_{2s} < 0.828$ by using the results of [(6),(7)].

Our analysis is very simple and elementary. We introduce the proposed results using the E.J. Candès idea, the T. Cai et al. idea, the Q. Mo and S. Li idea and the T. Cai and A. Zhang idea. We regard Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.5 as the main results in this paper. Otherwise, in Section 2, we prepare some notions and lemmas to prove main theorems. In Section 3, we introduce new bounds of δ_s and $\delta_k (k > s)$.

2 Preliminaries and Some Lemmas

In this section, we prepare some lemmas needed for the proofs of Theorem 3.1 and Theorem 3.3.

Lemma 2.1. Let k be a natural number with $k < n$. Then A is k-restrictly invertible if and only if $\tilde{A} \equiv \frac{A}{r_k}$ obeys the RIP of order k. If this is true, then

$$\tilde{\delta}_k = 1 - \frac{1}{(r_k w_k)^2}$$

and

$$\left(1 - \tilde{\delta}_k\right) \|\boldsymbol{a}\|_2^2 \le \|\tilde{A}\boldsymbol{a}\|_2^2 \le \|\boldsymbol{a}\|_2^2,$$

equivalently

$$r_k^2 \left(1 - \tilde{\delta}_k\right) \|\boldsymbol{a}\|_2^2 \le \|A\boldsymbol{a}\|_2^2 \le r_k^2 \|\boldsymbol{a}\|_2^2$$

for all k-sparse vector $\boldsymbol{a}$ in $\boldsymbol{R}^n$, where $\tilde{\delta}_k$ is the restricted isometry constant for $\tilde{A}$ and

$$w_k \equiv \min\left\{\| \left(A_T^\star A_T\right)^{-\frac{1}{2}} \|;\ T \subset \{1, 2, \cdots, n\} \text{ with } |T| = k\right\}.$$

Proof. Suppose that A is k-restrictly invertible. Then we have

$$\frac{1}{w_k}\|\boldsymbol{a}\|_2 \le \|A\boldsymbol{a}\|_2 \le r_k \|\boldsymbol{a}\|_2$$

for all k-sparse vector $\boldsymbol{a}$ in $\boldsymbol{R}^n$. Hence, $r_k w_k \ge 1$ and

$$\left(1 - \left(1 - \frac{1}{(r_k w_k)^2}\right)\right) \|\boldsymbol{a}\|_2^2 \le \|\tilde{A}\boldsymbol{a}\|_2^2 \le \|\boldsymbol{a}\|_2^2$$

for all k-sparse vector $\boldsymbol{a}$, which implies that $\tilde{A}$ obeys the RIP of order k and

$$\tilde{\delta}_k \le 1 - \frac{1}{(r_k w_k)^2}. \tag{2.1}$$

Conversely, suppose that $\tilde{A}$ obeys the RIP of order k. Then, since

$$\left(1 - \tilde{\delta}_k\right) \|\boldsymbol{a}\|_2^2 \le \|\tilde{A}\boldsymbol{a}\|_2^2 \le \|\boldsymbol{a}\|_2^2$$

for all k-sparse vector $\boldsymbol{a}$, it follows that A is k-restrictly invertible and $\tilde{\delta} \ge 1 - \frac{1}{(r_k w_k)^2}$, which implies by (2.1) that $\tilde{\delta}_k = 1 - \frac{1}{(r_k w_k)^2}$. This completes the proof.

Lemma 2.2. Take any $t \geq 1$ and positive integers s', s'' such that ts'' is an integer. Suppose that A is $(s'+s'')$-restrictly invertible. Then,

$$| < \tilde{A}\boldsymbol{a}', \tilde{A}\boldsymbol{a}'' > | \leq \frac{1}{2\sqrt{t}}\tilde{\delta}_{s'+s''}\|\boldsymbol{a}'\|_2\|\boldsymbol{a}''\|_2 \tag{2.2}$$

for any vectors $\boldsymbol{a}', \boldsymbol{a}'' \in \boldsymbol{R}^n$ with disjoint supports and sparsity ts'' and s', respectively.

Proof. We make use of the square root lifting inequality (10);

$$\theta_{s',ts''} \leq \frac{1}{\sqrt{t}}\theta_{s',s''}, \quad t \geq 1, \tag{2.3}$$

where $\theta_{k,k'}$ is the k,k'-restricted orthogonality constant. The k,k'-restricted orthogonality constant is the smallest number that satisfies

$$| < A\boldsymbol{c}, A\boldsymbol{c}' > | \leq \theta_{k,k'}\|\boldsymbol{c}\|_2\|\boldsymbol{c}'\|_2$$

for all k-sparse vector $\boldsymbol{c}$ and k'-sparse vector $\boldsymbol{c}'$ with disjoint supports. Take arbitrary s'-sparse vector $\boldsymbol{a}'$ and s''-sparse vector $\boldsymbol{a}''$ with disjoint supports. Then we show

$$| < \tilde{A}\boldsymbol{a}', \tilde{A}\boldsymbol{a}'' > | \leq \frac{\tilde{\delta}_{s'+s''}}{2}\|\boldsymbol{a}'\|_2\|\boldsymbol{a}''\|_2. \tag{2.4}$$

Indeed, it is sufficient to show this inequality without loss of generality in case $\|\boldsymbol{a}'\|_2 = \|\boldsymbol{a}''\|_2 = 1$. Since $\tilde{A}$ obeys the RIP of order $(s'+s'')$ and $r_{s'+s''}(\tilde{A}) \leq 1$, it follows that

$$\begin{aligned} 4 < \tilde{A}\boldsymbol{a}', \tilde{A}\boldsymbol{a}'' > &= \|\tilde{A}(\boldsymbol{a}'+\boldsymbol{a}'')\|_2^2 - \|\tilde{A}(\boldsymbol{a}'-\boldsymbol{a}'')\|_2^2 \\ &\leq \|\boldsymbol{a}'+\boldsymbol{a}''\|_2^2 - (1-\tilde{\delta}_s)\|\boldsymbol{a}'-\boldsymbol{a}''\|_2^2 \\ &= 2\tilde{\delta}_{s'+s''} \end{aligned}$$

and

$$\begin{aligned} 4 < \tilde{A}\boldsymbol{a}', \tilde{A}\boldsymbol{a}'' > &\geq (1-\tilde{\delta}_s)\|\boldsymbol{a}'+\boldsymbol{a}''\|_2^2 - \|\boldsymbol{a}'-\boldsymbol{a}''\|_2^2 \\ &= -2\tilde{\delta}_{s'+s''}, \end{aligned}$$

which implies that

$$| < \tilde{A}\boldsymbol{a}', \tilde{A}\boldsymbol{a}'' > | \leq \frac{1}{2}\tilde{\delta}_{s'+s''}.$$

Hence we have

$$\theta_{s',s''} \leq \frac{1}{2}\tilde{\delta}_{s'+s''}. \tag{2.5}$$

By (2.3) and (2.5) we have

$$\begin{aligned} | < \tilde{A}\boldsymbol{a}', \tilde{A}\boldsymbol{a}'' > | &\leq \theta_{ts'',s'}\|\boldsymbol{a}'\|_2\|\boldsymbol{a}''\|_2 \\ &\leq \frac{1}{\sqrt{t}}\theta_{s'',s'}\|\boldsymbol{a}'\|_2\|\boldsymbol{a}''\|_2 \\ &\leq \frac{1}{2\sqrt{t}}\tilde{\delta}_{s'+s''}\|\boldsymbol{a}'\|_2\|\boldsymbol{a}''\|_2 \end{aligned}$$

for all ts''-sparse vector $\boldsymbol{a}'$ and s'-sparse vector $\boldsymbol{a}''$ with disjoint supports.

Lemma 2.3. For any $\boldsymbol{a} \in \boldsymbol{R}^k$, we have

$$\|\boldsymbol{a}\|_2 \leq \frac{1}{\sqrt{k}}\|\boldsymbol{a}\|_1 + \frac{\sqrt{k}}{4}\left(\max_{1\leq i\leq k}|a_i| - \min_{1\leq i\leq k}|a_i|\right). \tag{2.6}$$

Proof. The proof of this lemma can be obtained by [(4), Proposition 2.1].

Suppose x is an original signal we need to recover and $x^\star$ is the solution of CS optimization problem (1.2). Let $\boldsymbol{h} \equiv \boldsymbol{x}^\star - \boldsymbol{x}$ and $\boldsymbol{h} = (h_1, \cdots, h_n)$. For simplicity, we assume that the index of $\boldsymbol{h}$ is sorted by $|h_1| \geq |h_2| \geq \cdots \geq |h_n|$. Throughout this paper, let T_0 be an arbitrary location of $\{1, 2, \cdots, n\}$ with $|T_0| = s$ and let $\{T_1, T_2, \cdots, T_l\}$ be a decomposition of $\{1, 2, \cdots, n\}$ with $|T_1| = s$, $|T_k| = s'$ $(2 \leq k \leq l-1)$ and $1 \leq |T_l| \equiv r \leq s'$, where $|T|$ is number of elements of T. We consider the decomposition of $\boldsymbol{h}$ as follows:

$$\begin{aligned}
\boldsymbol{h}_{T_1} &= (h_1^{(T_1)}, h_2^{(T_1)}, \cdots, h_s^{(T_1)}, 0, \cdots, 0) \\
\boldsymbol{h}_{T_2} &= (0, \cdots, 0, h_1^{(T_2)}, \cdots, h_{s'}^{(T_2)}, 0, \cdots, 0) \\
&\vdots \\
\boldsymbol{h}_{T_{l-1}} &= (0, \cdots, 0, h_1^{(T_{l-1})}, \cdots, h_{s'}^{(T_{l-1})}, 0, \cdots, 0) \\
\boldsymbol{h}_{T_l} &= (0, \cdots 0, h_1^{(T_l)}, \cdots, h_r^{(T_l)}).
\end{aligned}$$

This is due to the T. Cai et al. idea (4) in case of $s = s'$. We have the following Lemma 2.4–Lemma 2.10 for the decomposition $(\boldsymbol{h}_{T_1}, \boldsymbol{h}_{T_2}, \cdots, \boldsymbol{h}_{T_l})$ of $\boldsymbol{h}$. By definition of CS optimization (1.2), we have the following

Lemma 2.4. We have

$$\|\boldsymbol{h}_{T_0^c}\|_1 \leq 2\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + \|\boldsymbol{h}_{T_0}\|_1. \tag{2.7}$$

Refer to (11) for the proof of Lemma 2.3. T. Cai et al. have obtained a similar result for the location T_1.

Lemma 2.5. For $|T_0| = |T_1| = s$, we have

$$\|\boldsymbol{h}_{T_1^c}\|_1 \leq 2\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + \|\boldsymbol{h}_{T_1}\|_1. \tag{2.8}$$

Proof. Since $|T_0^c \cap T_1| = |T_0 \cap T_1^c|$, we have $\|\boldsymbol{h}_{T_0 \cap T_1^c}\|_1 \leq \|\boldsymbol{h}_{T_0^c \cap T_1}\|_1$, which implies by (2.7) that

$$\begin{aligned}
\|\boldsymbol{h}_{T_1^c}\|_1 &= \|\boldsymbol{h}_{T_0 \cap T_1^c}\|_1 + \|\boldsymbol{h}_{T_0^c}\|_1 - \|\boldsymbol{h}_{T_1 \cap T_0^c}\|_1 \\
&\leq 2\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + \|\boldsymbol{h}_{T_1}\|_1 \\
&\quad +2\left(\|\boldsymbol{h}_{T_0 \cap T_1^c}\|_1 - \|\boldsymbol{h}_{T_1 \cap T_0^c}\|_1\right) \\
&\leq 2\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + \|\boldsymbol{h}_{T_1}\|_1.
\end{aligned}$$

Lemma 2.6. We have

$$\begin{aligned}
\sum_{i \geq 2} \|\boldsymbol{h}_{T_i}\|_2 \quad &\leq \quad \frac{2}{\sqrt{s'}}\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 \\
&\quad + \left(\frac{\sqrt{s}}{\sqrt{s'}} + \frac{\sqrt{s'}}{4\sqrt{s}}\right)\|\boldsymbol{h}_{T_1}\|_2.
\end{aligned} \tag{2.9}$$

Proof. By using Lemma 2.3, we have

$$\begin{aligned}
\|\boldsymbol{h}_{T_i}\|_2 \leq \frac{1}{\sqrt{s'}}\|\boldsymbol{h}_{T_i}\|_1 \quad &+ \quad \frac{\sqrt{s'}}{4}\left(|h_1^{(T_i)}| - |h_1^{(T_{i+1})}|\right) \\
&, 3 \leq i \leq l-1,
\end{aligned}$$

which implies by Lemma 2.4 that

$$\begin{aligned}\sum_{i\geq 2}\|\boldsymbol{h}_{T_i}\|_2 &\leq \frac{1}{\sqrt{s'}}\sum_{i\geq 2}\|\boldsymbol{h}_{T_i}\|_1+\frac{\sqrt{s'}}{4}|h_1^{(T_2)}| \\ &\leq \frac{2}{\sqrt{s'}}\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1+\left(\frac{\sqrt{s}}{\sqrt{s'}}+\frac{\sqrt{s'}}{4\sqrt{s}}\right)\|\boldsymbol{h}_{T_1}\|_2.\end{aligned} \tag{2.10}$$

Similarly we have the following

Lemma 2.7. Let $s'<s$. We consider the decomposition $T_1=\{T_1',T_1''\}$ of T_1 with $|T_1'|=s'$ and $|T_1''|=s''$. Then, $s'=(1-t)s,\ s''=ts$ for some $t\in(0,1)$ and

$$\begin{aligned}\sum_{i\geq 2}\|\boldsymbol{h}_{T_i}\|_2 &\leq \frac{2}{\sqrt{s(1-t)}}\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 \\ &\quad+\left(\frac{1}{\sqrt{1-t}}+\frac{\sqrt{1-t}}{4}\right)\|\boldsymbol{h}_{T_1}\|_2.\end{aligned} \tag{2.11}$$

We put $\|\boldsymbol{h}_{T_2}\|_1\equiv p\sum_{i\geq 2}\|\boldsymbol{h}_{T_i}\|_1=p\|\boldsymbol{h}_{T_1^c}\|_1$. Then $0\leq p\leq 1$ and $\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_1=(1-p)\|\boldsymbol{h}_{T_1^c}\|_1$. Then the following Lemma 2.8 is easily shown and Lemma 2.9 is also easily shown by using the inequality (2.10).

Lemma 2.8. We have

$$\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_2^2<\frac{p(1-p)}{s'}\|\boldsymbol{h}_{T_1^c}\|_1^2. \tag{2.12}$$

Proof. This follows from

$$\begin{aligned}\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_2^2 &\leq |h_1^{(T_3)}|\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_1 \\ &\leq \frac{1}{s'}\|\boldsymbol{h}_{T_2}\|_1\left(\sum_{i\geq 2}\|\boldsymbol{h}_{T_i}\|_1-\|\boldsymbol{h}_{T_2}\|_1\right) \\ &= \frac{p}{s'}(1-p)\|\boldsymbol{h}_{T_1^c}\|_1^2.\end{aligned}$$

Lemma 2.9. We have

$$\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_2<\frac{1-3p/4}{\sqrt{s'}}\|\boldsymbol{h}_{T_1^c}\|_1. \tag{2.13}$$

Proof. By (2.10), we have

$$\begin{aligned}\sqrt{s'}\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_2 &\leq \sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_1+\frac{s'}{4}|h_1^{(T_3)}| \\ &\leq \sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_1+\frac{1}{4}\|\boldsymbol{h}_{T_2}\|_1 \\ &= \left(1-\frac{3}{4}p\right)\|\boldsymbol{h}_{T_1^c}\|_1.\end{aligned}$$

Lemma 2.10. (i) Let $s' \leq s$. Suppose that A is $(s+s')$-restrictly invertible. Then,

$$\begin{aligned}\|\sum_{i\geq 3}\tilde{A}\boldsymbol{h}_{T_i}\|_2^2 &\leq \frac{1}{2s'}\left((2-\delta_{s+s'})\,p(1-p)\right.\\ &\quad\left.+\delta_{s+s'}\left(1-\frac{3}{4}p\right)^2\right)\|\boldsymbol{h}_{T_1^c}\|_1^2. \qquad (2.14)\end{aligned}$$

(ii) Let $s' > s$. Suppose that A is $2s'$-restrictly invertible. Then,

$$\begin{aligned}\|\sum_{i\geq 3}\tilde{A}\boldsymbol{h}_{T_i}\|_2^2 &\leq \frac{1}{2s'}\left((2-\delta_{2s'})\,p(1-p)\right.\\ &\quad\left.+\delta_{2s'}\left(1-\frac{3}{4}p\right)^2\right)\|\boldsymbol{h}_{T_1^c}\|_1^2. \qquad (2.15)\end{aligned}$$

Proof. (i) Since $\tilde{A}$ obeys the RIP of order $(s+s')$ and $r_{2s'}(A) \leq 1$, it follows from Lemma 2.2, Lemma 2.7 and Lemma 2.8 that

$$\begin{aligned}&\|\sum_{i\geq 3}\tilde{A}\boldsymbol{h}_{T_i}\|_2^2\\ =&\sum_{i\geq 3}\|\tilde{A}\boldsymbol{h}_{T_i}\|_2^2+2\sum_{3\leq i<j\leq l}<\tilde{A}\boldsymbol{h}_{T_i},\tilde{A}\boldsymbol{h}_{T_j}>\\ \leq&\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_2^2+2\sum_{3\leq i<j\leq l}\frac{\tilde{\delta}_{2s'}}{2}\|\boldsymbol{h}_{T_i}\|_2\|\boldsymbol{h}_{T_j}\|_2\\ \leq&\sum_{i\geq 3}\left(1-\frac{\tilde{\delta}_{s+s'}}{2}\right)\|\boldsymbol{h}_{T_i}\|_2^2+\frac{\tilde{\delta}_{s+s'}}{2}\left(\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_2^2\right.\\ &\left.+2\sum_{3\leq i<j\leq l}\frac{\tilde{\delta}_{2s'}}{2}\|\boldsymbol{h}_{T_i}\|_2\|\boldsymbol{h}_{T_j}\|_2\right)\\ =&\left(1-\frac{\tilde{\delta}_{s+s'}}{2}\right)\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_2^2+\frac{\tilde{\delta}_{s+s'}}{2}\left(\sum_{i\geq 3}\|\boldsymbol{h}_{T_i}\|_2\right)^2\\ \leq&\left(1-\frac{\tilde{\delta}_{s+s'}}{2}\right)\frac{p(1-p)}{s'}\|\boldsymbol{h}_{T_i^c}\|_1^2\\ &+\frac{\tilde{\delta}_{s+s'}}{2s'}\left(1-\frac{3}{4}p\right)^2\|\boldsymbol{h}_{T_i^c}\|_1^2\\ =&\frac{1}{2s'}\left(\left(2-\tilde{\delta}_{s+s'}\right)p(1-p)+\tilde{\delta}_{s+s'}\left(1-\frac{3}{4}p\right)^2\right)\|\boldsymbol{h}_{T_i^c}\|_1^2.\end{aligned}$$

(2.16)

(ii) This is shown similarly to (i).

3 Main results

In this section, we introduce the main results of the sufficient condition of $\delta_k (k \geq s)$ under the assumption that A is restrictly invertible.

3.1 Bound for $\tilde{\delta}_s$

We have established the sufficient condition $\tilde{\delta}_s$ for CS optimization problem in Theorem 3.1 and Theorem 3.2.

Theorem 3.1. Assume that A is s-restrictly invertible and $\tilde{\delta}_s < \frac{2}{2+\sqrt{5}} \approx 0.472$. Then, the solution $\boldsymbol{x}^\star$ to (1.2) obeys

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \le C_0 \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + C_1 \varepsilon, \tag{3.1}$$

where

$$C_0 = \frac{9\tilde{\delta}_s}{2\sqrt{5s}\left(1 - \frac{\sqrt{5}+2}{2}\tilde{\delta}_s\right)}, \quad C_1 = \frac{2}{r_s\left(1 - \frac{\sqrt{5}+2}{2}\tilde{\delta}_s\right)}.$$

Proof. Let $\{T_1, T_2, \cdots, T_l\}$ be a decomposition of $\{1, 2, \cdots, n\}$ with $|T_1| = s$, $|T_k| = s' < s(2 \le k \le l-1)$ and $1 \le |T_l| \equiv r \le s'$. We consider the decomposition $T_1 = \{T_1', T_1''\}$ of T_1 with $|T_1'| = s'$ and $|T_1''| = s''$. Then, $s' = (1-t)s$ and $s'' = ts$ for some $t \in (0,1)$. Since $\tilde{A} \equiv \frac{A}{r_s}$ obeys the RIP of order $s = s' + s'' = \frac{1}{t}s''$, it follows from Lemma 2.2 that

$$| < \tilde{A}\boldsymbol{h}_{T_1}, \tilde{A}\boldsymbol{h}_{T_j} > | \le \frac{1}{2\sqrt{t}}\tilde{\delta}_s \|\boldsymbol{h}_{T_1}\|_2 \|\boldsymbol{h}_{T_j}\|_2, \quad j \ge 2,$$

which implies by $r_s(\tilde{A}) \le 1$ that

$$\begin{aligned}
(1 - \tilde{\delta}_s)\|\boldsymbol{h}_{T_1}\|_2^2 &\le < \tilde{A}\boldsymbol{h}_{T_1}, \tilde{A}\boldsymbol{h} - \sum_{j\ge 2} \tilde{A}\boldsymbol{h}_{T_j} > \\
&\le 2\|\tilde{A}\boldsymbol{h}_{T_1}\|_2 \varepsilon + \sum_{j \ge 2} | < \tilde{A}\boldsymbol{h}_{T_1}, \tilde{A}\boldsymbol{h}_{T_j} > | \\
&\le \frac{2}{r_s}\varepsilon \|\boldsymbol{h}_{T_1}\|_2 \\
&\quad + \frac{1}{2\sqrt{t}}\tilde{\delta}_s \|\boldsymbol{h}_{T_1}\|_2 \left(\sum_{j \ge 2} \|\boldsymbol{h}_{T_j}\|_2 \right).
\end{aligned}$$

Thus, by Lemma 2.7 and the above inequality, we have

$$\begin{aligned}
(1 - \tilde{\delta}_s)\|\boldsymbol{h}_{T_1}\|_2 &\le \frac{2}{r_s}\varepsilon + \frac{\tilde{\delta}_s}{\sqrt{(1-t)ts}} \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 \\
&+ \frac{1}{2\sqrt{t}} \left(\frac{1}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{4} \right) \tilde{\delta}_s \|\boldsymbol{h}_{T_1}\|_2.
\end{aligned} \tag{3.2}$$

Here, put $f(t) = \frac{1}{\sqrt{t}}\left(\frac{1}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{4}\right)$. Then, f is increasing when $\frac{5}{9} < t < 1$ and decreasing when $0 < t < \frac{5}{9}$. Thus, when $t = \frac{5}{9}$, we have

$$\begin{aligned}
(1 - \tilde{\delta}_s)\|\boldsymbol{h}_{T_1}\|_2 &\le \frac{2}{r_s}\varepsilon + \frac{9}{2\sqrt{5s}}\tilde{\delta}_s \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 \\
&\quad + \frac{\sqrt{5}}{2}\tilde{\delta}_s \|\boldsymbol{h}_{T_1}\|_2,
\end{aligned} \tag{3.3}$$

so that by assumption $\tilde{\delta}_s < \frac{2}{2+\sqrt{5}} \approx 0.472$,

$$\|\boldsymbol{h}_{T_1}\|_2 \le \frac{1}{1 - (\sqrt{5}+1)\tilde{\delta}_s} \left(\frac{2}{r_s}\varepsilon + \frac{9}{2\sqrt{5s}}\tilde{\delta}_s \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 \right). \tag{3.4}$$

Furthermore, it follows from Lemma 2.7 that

$$\|\boldsymbol{h}_{T_1^c}\|_2 \le \sum_{j\ge 2}\|\boldsymbol{h}_{T_j}\|_2 \le \frac{3}{\sqrt{s}}\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + \frac{5}{3}\|\boldsymbol{h}_{T_1}\|_2, \tag{3.5}$$

which implies by (3.4) that

$$\|\boldsymbol{x}-\boldsymbol{x}^\star\|_2 \le \|\boldsymbol{h}_{T_1}\|_2 + \|\boldsymbol{h}_{T_1^c}\|_2 \le C_0\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + C_1\varepsilon.$$

This completes the proof.

By using Theorem 3.3 in (6), we obtain a better result than that of Theorem 3.1.

Theorem 3.2. Assume that A is s-restrictly invertible and $\tilde{\delta}_s < \frac{1}{2} = 0.5$. Then, the solution $\boldsymbol{x}^\star$ to (1.2) obeys

$$\|\boldsymbol{x}-\boldsymbol{x}^\star\|_2 \le D_0\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + D_1\varepsilon,$$

where

$$\begin{aligned} D_0 &= \frac{2\left(\sqrt{\tilde{\delta}_s(1-2\tilde{\delta}_s)} - (2-\sqrt{2})\tilde{\delta}_s + 1\right)}{(1-2\tilde{\delta}_s)\sqrt{s}}, \\ D_1 &= \frac{2\sqrt{2}}{(1-2\tilde{\delta}_s)r_s}. \end{aligned}$$

Proof. Since A is s-restrictly invertible, we have

$$\frac{1}{w_s}\|\boldsymbol{a}\|_2 \le \|A\boldsymbol{a}\|_2 \le r_s\|\boldsymbol{a}\|_2$$

for all s-sparse vectors $\boldsymbol{a}$ in $\boldsymbol{R}^n$. We here put $\hat{A} \equiv A/\frac{\sqrt{(r_s w_s)^2+1}}{\sqrt{2}w_s}$ and $\hat{\delta}_s = \frac{(r_s w_s)^2-1}{(r_s w_s)^2+1}$. Then we have

$$(1-\hat{\delta}_s)\|\boldsymbol{a}\|_2^2 \le \|\hat{A}\boldsymbol{a}\|_2^2 \le (1+\hat{\delta}_s)\|\boldsymbol{a}\|_2^2$$

for all s-sparse vectors $\boldsymbol{a}$ in $\boldsymbol{R}^n$, that is, $\hat{A}$ obeys the RIP of order s with the restricted isometry constant $\hat{\delta}_s$. Furthermore, by (2.1) we have

$$\tilde{\delta}_s = 1 - \frac{1}{(r_s w_s)^2} < \frac{1}{2},$$

which implies

$$\hat{\delta}_s = 1 - \frac{2}{1+(r_s w_s)^2} < \frac{1}{3}.$$

Considering the following equality (3.6) instead of (1.1):

$$\hat{\boldsymbol{y}} = \hat{A}\boldsymbol{x} + \hat{\boldsymbol{z}}, \tag{3.6}$$

where $\hat{\boldsymbol{y}} = \frac{\boldsymbol{y}}{\sqrt{q}}$, $\hat{A} = \frac{A}{\sqrt{q}}$ and $\hat{\boldsymbol{z}} = \frac{\boldsymbol{z}}{\sqrt{q}}$ (here, q denotes $\frac{\sqrt{(r_s w_s)^2+1}}{\sqrt{2}w_s}$), it follows from Theorem 3.3 in (6) that

$$\begin{aligned} \|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 &\le \frac{2\sqrt{2}\left(2\hat{\delta}_s + \sqrt{(1-3\hat{\delta}_s)\hat{\delta}_s}\right) + 2(1-3\hat{\delta}_s)}{1-3\hat{\delta}_s}\frac{\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1}{\sqrt{s}} \\ &\quad + \frac{2\sqrt{2(1+\hat{\delta}_s)}}{1-3\hat{\delta}_s}\frac{\varepsilon}{q} \\ &= D_0\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + D_1\varepsilon. \end{aligned}$$

This completes the proof.

3.2 Bound for $\tilde{\delta}_k$ $(s < k)$

Using the E.J. Candès decomposition $\{T_1, T_2, \cdots, T_q\}$ of T_0^c with $|T_k| = s$ $(k = 1, \cdots, q)$ and $|h_1^{(T_1)}| \geq |h_2^{(T_1)}| \geq \cdots \geq |h_s^{(T_1)}| \geq |h_1^{(T_2)}| \geq |h_2^{(T_2)}| \geq \cdots$, Q. Mo and S. Li have obtained a new bound of the isometry constant δ_{2s} (5). In Theorem 3.3, using the decomposition $\{T_1, T_2, \cdots, T_l\}$ of $\{1, 2, \cdots, n\}$ stated in Section 2 and taking an arbitrary natural number s', we have obtained a bound of the isometry constant $\tilde{\delta}_k$ $(s < k)$ under the assumption that A is k-restrictly invertible..

Theorem 3.3. (i) Let $\frac{1}{8}s \leq s' \leq s$. We assume A is $(s + s')$-restrictly invertible and $\frac{\sqrt{s}}{\sqrt{s'}}\alpha_{s,s'} < 1$, where

$$\alpha_{s,s'} = \sqrt{\frac{4(2 + 3\tilde{\delta}_{s+s'} - 4\tilde{\delta}_{s+s'}^2)}{(1 - \tilde{\delta}_{s+s'})(64 - 57\tilde{\delta}_{s+s'})}}.$$

Then,

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq E_0\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + E_1\varepsilon, \tag{3.7}$$

where

$$\begin{aligned} E_0 &= \frac{2\left(1 + \left(1 + \frac{\sqrt{s'}}{4\sqrt{s}}\right)\alpha_{s,s'}\right)}{\sqrt{s'}\left(1 - \sqrt{\frac{s}{s'}}\alpha_{s,s'}\right)}, \\ E_1 &= \frac{2\sqrt{2}\left(1 + \sqrt{\frac{s}{s'}} + \frac{\sqrt{s'}}{4\sqrt{s}}\right)}{r_{s+s'}\sqrt{1 - \tilde{\delta}_{s+s'}}\left(1 - \sqrt{\frac{s}{s'}}\alpha_{s,s'}\right)}. \end{aligned}$$

(ii) Let $s' \geq s$. We assume that A is $2s'$-restrictly invertible and $\frac{\sqrt{s}}{\sqrt{s'}}\alpha_{s'} < 1$, where

$$\alpha_{s'} = \sqrt{\frac{4(2 + 3\tilde{\delta}_{2s'} - 4\tilde{\delta}_{2s'}^2)}{(1 - \tilde{\delta}_{2s'})(64 - 57\tilde{\delta}_{2s'})}}.$$

Then,

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq E_0'\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + E_1'\varepsilon, \tag{3.8}$$

where

$$\begin{aligned} E_0' &= \frac{2\left(1 + \left(1 + \frac{\sqrt{s'}}{4\sqrt{s}}\right)\alpha_{s'}\right)}{\sqrt{s'}\left(1 - \sqrt{\frac{s}{s'}}\alpha_{s'}\right)}, \\ E_1' &= \frac{2\sqrt{2}\left(1 + \sqrt{\frac{s}{s'}} + \frac{\sqrt{s'}}{4\sqrt{s}}\right)}{r_{2s'}\sqrt{1 - \tilde{\delta}_{2s'}}\left(1 - \sqrt{\frac{s}{s'}}\alpha_{s'}\right)}. \end{aligned}$$

Proof. (i) Let $\frac{1}{8}s \leq s' \leq s$. By the definition of RIP and Lemma 2.10, we have

$$\begin{aligned}
& \left(1-\tilde{\delta}_{s+s'}\right) \|\boldsymbol{h}_{T_1}\|_2^2 \\
= \;& (1-\tilde{\delta}_{s+s'})\|\boldsymbol{h}_{T_1\cup T_2}\|_2^2 - (1-\tilde{\delta}_{s+s'})\|\boldsymbol{h}_{T_2}\|_2^2 \\
\leq \;& \|\tilde{A}\boldsymbol{h}_{T_1\cup T_2}\|_2^2 - (1-\tilde{\delta}_{s+s'})\|\boldsymbol{h}_{T_2}\|_2^2 \\
\leq \;& \|\tilde{A}\boldsymbol{h} - \sum_{j\geq 3}\tilde{A}\boldsymbol{h}_{T_j}\|_2^2 - \frac{(1-\tilde{\delta}_{s+s'})}{s'}p^2\|\boldsymbol{h}_{T_1^c}\|_1^2 \\
\leq \;& \left(\frac{2}{r_{s+s'}}\varepsilon + \|\sum_{j\geq 3}\tilde{A}\boldsymbol{h}_{T_j}\|_2\right)^2 - \frac{(1-\tilde{\delta}_{s+s'})}{s'}p^2\|\boldsymbol{h}_{T_1^c}\|_1^2 \\
\leq \;& \frac{4}{r_{s+s'}^2}\varepsilon^2 + \frac{4}{r_{s+s'}}\varepsilon\frac{1}{\sqrt{2s'}} \\
\times \;& \sqrt{(2-\tilde{\delta}_{s+s'})p(1-p) + \tilde{\delta}_{s+s'}\left(1-\frac{3}{4}p\right)^2}\|\boldsymbol{h}_{T_1^c}\|_1 \\
+ \;& \frac{1}{2s'}\left((2-\tilde{\delta}_{s+s'})p(1-p) + \tilde{\delta}_{s+s'}\left(1-\frac{3}{4}p\right)^2\right. \\
& \left.-2(1-\tilde{\delta}_{s+s'})p^2\right)\|\boldsymbol{h}_{T_1^c}\|_1^2.
\end{aligned}$$

Since

$$\begin{aligned}
& \sqrt{(2-\tilde{\delta}_{s+s'})p(1-p) + \tilde{\delta}_{s+s'}\left(1-\frac{3}{4}p\right)^2} \\
\leq \;& \sqrt{\frac{8(2-\tilde{\delta}_{s+s'})}{32-25\tilde{\delta}_{s+s'}}},
\end{aligned}$$

$$\begin{aligned}
& (2-\tilde{\delta}_{s+s'})p(1-p) + \tilde{\delta}_{s+s'}\left(1-\frac{3}{4}p\right)^2 \\
& -2(1-\tilde{\delta}_{s+s'})p^2 \leq \frac{8(2+3\tilde{\delta}_{s+s'}-4\tilde{\delta}_{s+s'}^2)}{64-57\tilde{\delta}_{s+s'}}
\end{aligned}$$

and

$$\frac{8(2-\tilde{\delta}_{s+s'})}{32-25\tilde{\delta}_{s+s'}} \leq 2\frac{8(2+3\tilde{\delta}_{s+s'}-4\tilde{\delta}_{s+s'}^2)}{64-57\tilde{\delta}_{s+s'}},$$

we have

$$\begin{aligned}
& (1-\tilde{\delta}_{s+s'})\|\boldsymbol{h}_{T_1}\|_2^2 \\
\leq \;& \left(\frac{2\sqrt{2}}{r_{s+s'}}\varepsilon + \sqrt{\frac{1}{2s'}}\sqrt{\frac{8(2+3\tilde{\delta}_{s+s'}-4\tilde{\delta}_{s+s'}^2)}{64-57\tilde{\delta}_{s+s'}}}\|\boldsymbol{h}_{T_1^c}\|_1\right)^2,
\end{aligned}$$

which implies by Lemma 2.4 that

$$\|\boldsymbol{h}_{T_1}\|_2 \leq \frac{2\sqrt{2}}{r_{s+s'}\sqrt{1-\tilde{\delta}_{s+s'}}}\varepsilon + \frac{\alpha_{s,s'}}{\sqrt{s'}}\left(2\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + \|\boldsymbol{h}_{T_1}\|_1\right).$$

By the assumption $\frac{\sqrt{s}}{\sqrt{s'}}\alpha_{s,s'} < 1$, we have

$$\begin{aligned}\|\boldsymbol{h}_{T_1}\|_2 &\leq \frac{2\sqrt{2}}{r_{s+s'}\sqrt{1-\tilde{\delta}_{s+s'}}\left(1-\sqrt{\frac{s}{s'}}\alpha_{s,s'}\right)}\varepsilon \\ &\quad + \frac{2\alpha_{s,s'}}{\sqrt{s'}\left(1-\sqrt{\frac{s}{s'}}\alpha_{s,s'}\right)}\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1, \end{aligned} \tag{3.9}$$

which implies by Lemma 2.6 that

$$\begin{aligned}\|\boldsymbol{x}-\boldsymbol{x}^\star\|_2 &\leq \|\boldsymbol{h}_{T_1}\|_2 + \sum_{j\geq 2}\|\boldsymbol{h}_{T_j}\|_2 \\ &\leq \left(1+\sqrt{\frac{s}{s'}}+\frac{\sqrt{s'}}{4\sqrt{s}}\right)\|\boldsymbol{h}_{T_1}\|_2 \\ &\quad +\frac{2}{\sqrt{s'}}\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 \\ &\leq \frac{2\left(1+\left(1+\frac{\sqrt{s'}}{4\sqrt{s}}\right)\alpha_{s,s'}\right)}{\sqrt{s'}\left(1-\sqrt{\frac{s}{s'}}\alpha_{s,s'}\right)}\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 \\ &\quad + \frac{2\sqrt{2}\left(1+\sqrt{\frac{s}{s'}}+\frac{\sqrt{s'}}{4\sqrt{s}}\right)}{r_{s+s'}\sqrt{1-\tilde{\delta}_{s+s'}}\left(1-\sqrt{\frac{s}{s'}}\alpha_{s,s'}\right)}\varepsilon. \end{aligned} \tag{3.10}$$

(ii) This is shown similarly to (i).

We here find the bound of the restricted isometry $\tilde{\delta}.$ by the condition $\sqrt{\frac{s}{s'}}\alpha. < 1$ in Theorem 3.2. Let $p \equiv \frac{s'}{s} > \frac{1}{8}$. Then the equality

$$(57p+16)t^2 - (121p+12)t + 64p - 8 = 0, \quad 0 < t < 1 \tag{3.11}$$

has the unique solution t_p and the following hold:
When $\frac{1}{8} < p \leq 1$, the condition $\sqrt{\frac{s}{s'}}\alpha_{s,s'} < 1$ in Theorem 3.2, (i) holds if and only if

$$\delta_{(1+p)s} < t_p. \tag{3.12}$$

When $p \geq 1$, the condition $\sqrt{\frac{s}{s'}}\alpha_{s'} < 1$ in Theorem 3.2, (ii) holds if and only if

$$\delta_{2ps} < t_p. \tag{3.13}$$

Putting $p = 1$ in (3.12), we obtain the following result for bound for $\tilde{\delta}_{2s}$.

Corollary 3.4. We assume that A is $2s$-restrictly invertible and $\tilde{\delta}_{2s} < \frac{133-\sqrt{1337}}{146} \approx 0.661$. Then,

$$\|\boldsymbol{x}-\boldsymbol{x}^\star\|_2 \leq E_0\|\boldsymbol{x}-\boldsymbol{x}_{T_0}\|_1 + E_1\varepsilon,$$

where

$$\begin{aligned} E_0 &= \frac{2\left(1+\frac{5}{4}\alpha_s\right)}{\sqrt{s}(1-\alpha_s)}, \\ E_1 &= \frac{9\sqrt{2}}{2r_{2s}\sqrt{1-\delta_{2s}}(1-\alpha_s)}, \\ \alpha_s &= 2\sqrt{\frac{2+3\delta_{2s}-4\delta_{2s}^2}{(1-\delta_{2s})(64-57\delta_{2s})}}. \end{aligned}$$

By taking various numbers p, we can obtain some others $\tilde{\delta}_k (k > s)$. Here we give several bounds of the isometry constants $\tilde{\delta}_k$.

Example
(i) Let $p = \frac{2}{3}$. Then $t_{\frac{2}{3}} = 0.551$ and so $\tilde{\delta}_{\frac{5}{3}s} < 0.551.$
(ii) Let $p = \frac{3}{4}$. Then $t_{\frac{3}{4}} = 0.585$ and so $\tilde{\delta}_{\frac{7}{4}s} < 0.585.$
(iii) Let $p = \frac{3}{2}$. Then $t_{\frac{3}{2}} = 0.749$ and so $\tilde{\delta}_{3s} < 0.749.$
(iv) Let $p = 2$. Then $t_2 = 0.8$ and so $\tilde{\delta}_{4s} < 0.8.$

Using Theorem 2.1 in (7), we have the following

Theorem 3.5. Assume that for any $t \geq \frac{4}{3}$ A is ts-restrictly invertible and

$$\tilde{\delta}_{ts} < 2\sqrt{t-1}\left(\sqrt{t} - \sqrt{t-1}\right). \tag{3.14}$$

Then, the solution $\boldsymbol{x}^\star$ to (1.2) obeys

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq F_0 \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + F_1 \varepsilon,$$

where

$$\begin{aligned} F_0 &= \left(\frac{\sqrt{2}\tilde{\delta}_{ts} + \sqrt{t\left(\sqrt{\frac{t-1}{t}}(2-\tilde{\delta}_{ts}) - \tilde{\delta}_{ts}\right)\tilde{\delta}_{ts}}}{t\left(\sqrt{\frac{t-1}{t}}(2-\tilde{\delta}_{ts}) - \tilde{\delta}_{ts}\right)} + 1 \right) \frac{2}{\sqrt{s}} \\ F_1 &= \frac{8\sqrt{t(t-1)(2-\tilde{\delta}_{ts})(1+\tilde{\delta}_{ts})}}{t\left(\sqrt{\frac{t-1}{t}}(2-\tilde{\delta}_{ts}) - \tilde{\delta}_{ts}\right)} \frac{1}{r_{ts}}. \end{aligned}$$

Proof. This is proved similarly to Theorem 3.2. Since A is ts-restrictly invertible, $\hat{A} \equiv A / \frac{\sqrt{(r_{ts}w_{ts})^2+1}}{\sqrt{2}w_{ts}}$ obeys the RIP of order ts with the restricted isometry constant $\hat{\delta}_{ts} = \frac{(r_{ts}w_{ts})^2-1}{(r_{ts}w_{ts})^2+1}$. Then since

$$\tilde{\delta}_{ts} = 1 - \frac{1}{(r_{ts}w_{ts})^2} < 2\sqrt{t-1}(\sqrt{t} - \sqrt{t-1}),$$

it follows that $\hat{\delta}_{ts} < \sqrt{\frac{t-1}{t}}$, which implies by Theorem 2.1 in (7) that

$$\begin{aligned} \|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 &\leq \left(\frac{\sqrt{2}\hat{\delta}_{ts} + \sqrt{t\left(\sqrt{\frac{t-1}{t}} - \hat{\delta}_{ts}\right)\hat{\delta}_{ts}}}{t\left(\sqrt{\frac{t-1}{t}} - \hat{\delta}_{ts}\right)} + 1 \right) \frac{2}{\sqrt{s}} \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 \\ &\quad + \frac{4\sqrt{2t(t-1)(1+\hat{\delta}_{ts})}}{t\left(\sqrt{\frac{t-1}{t}} - \hat{\delta}_{ts}\right)} \frac{\sqrt{2}w_{ts}}{\sqrt{(r_{ts}w_{ts})^2+1}} \varepsilon \\ &= F_0 \|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + F_1 \varepsilon. \end{aligned}$$

This completes the proof.

Putting $t = 2$ in (3.14), we obtain the following result for bound for $\tilde{\delta}_{2s}$.

Corollary 3.6. We assume that A is $2s$-restrictly invertible and $\tilde{\delta}_{2s} < 2(\sqrt{2}-1) \approx 0.828$. Then

$$\|\boldsymbol{x} - \boldsymbol{x}\|_2 \le F_0\|\boldsymbol{x} - \boldsymbol{x}_{T_0}\|_1 + F_1\varepsilon,$$

where

$$\begin{aligned} F_0 &= \left(\frac{2(\sqrt{2}-\tilde{\delta}_{2s}) + \sqrt{\left(2\sqrt{2}-(2+\sqrt{2})\tilde{\delta}_{2s}\right)\tilde{\delta}_{2s}}}{2\sqrt{2}-(2+\sqrt{2})\tilde{\delta}_{2s}}\right)\frac{2}{\sqrt{s}}, \\ F_1 &= \frac{8\sqrt{2(2-\tilde{\delta}_{2s})(1+\tilde{\delta}_{2s})}}{2\sqrt{2}-(2+\sqrt{2})\tilde{\delta}_{2s}}\frac{1}{r_{2s}}. \end{aligned}$$

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