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The Topp Leone Generalized Inverted Kumaraswamy Distribution: Properties and Applications

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

We propose a new four parameters continuous model called the Topp Leone generalized inverted Kumaraswamy distribution which extends the generalized inverted Kumaraswamy distribution. Basic mathematical properties of the new distribution are invstigated such as; quantile function, raw and incomplete moments, generating functions, probability weighted moments, order statistics, Rényi entropy, stochastic ordering and stress strength model. The method of maximum likelihood is used to estimate the model parameters of the new distribution. A Monte Carlo simulation is conducted to examine the behavior of the parameter estimates. The applicability of the new model is demonstrated by means of two real applications.

Keywords: Inverted Kumaraswamy distribution, probability weighted moments, order statistics, stochastic ordering, stress strength model, Topp Leone-G family.

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1 Introduction

The inverted distributions have many applications in different fields such as; biological sciences, life testing problems, engineering sciences, environmental studies and econometrics. In the last two decades, the researchers proposed many inverted distributions due to its great applications; for example, [1] studied the inverted Burr type XII distribution, [2] introduced the inverted Pareto type I distribution, [3] studied the inverted exponential model and [4] proposed the exponentiated inverted Weibull distribution and [5] investigated the inverted Kumaraswamy distribution, among others.

Moreover, the inverted Kumaraswamy (IKw) distribution has probability density function (pdf) and distribution function (cdf) are given, respictevely, by

$$f_{IKw}(x;\alpha,\beta)=\alpha\beta\left(1+x\right)^{-\alpha-1}\left\{1-\left(1+x\right)^{-\alpha}\right\}^{\beta-1},x>0, \tag{1}$$

and

$$F_{IKw}(x;\alpha,\beta)=\left\{1-\left(1+x\right)^{-\alpha}\right\}^{\beta},x\geq 0, \tag{2}$$

where $\alpha>0$ and $\beta>0$ are two shape parameters. [6] introduced an extension of the IKw distribution called the generalized inverted Kumaraswamy (GIKw) distribution with pdf and cdf are given, respictevely, by

$$f_{GIKw}(x;\alpha,\beta,\lambda)=\alpha\beta\lambda x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1}\left\{1-\left(1+x^{\lambda}\right)^{-\alpha}\right\}^{\beta-1},\alpha,\beta,\lambda>0,x>0, \tag{3}$$

and

$$F_{GIKw}(x;\alpha,\beta,\lambda)=\left\{1-\left(1+x^{\lambda}\right)^{-\alpha}\right\}^{\beta},x\geq 0, \tag{4}$$

$g(x;\phi)$ and $G(x;\phi)$ be the pdf and cdf of a baseline model with parameter vector ϕ. [7] introduced the Topp Leone-G (TL-G) family of distributions with cdf and pdf given by

$$F_{TL-G}(x;\phi)=\left\{1-\bar{G}(x;\phi)^{2}\right\}^{\theta},\theta>0,x\in R, \tag{5}$$

And

$$f_{TL-G}(x;\phi)=2\theta\, g(x;\phi)\,\bar{G}(x;\phi)\left\{1-\bar{G}(x;\phi)^{2}\right\}^{\theta-1},x\in R, \tag{6}$$

where, $\bar{G}(x;\phi)=1-G(x;\phi)$.

This study aims to introduce a new generalization of the GIKw distribution by using the TL-G family called the Topp Leone generalized inverted Kumaraswamy (TLGIKw) distribution to obtain more flexability and accuracy in fitting data than some well known distributions. The cdf and pdf of the TLGIKw distribution are given, respectively, by

$$F(x;\alpha,\lambda,\beta,\theta)=\left[1-\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}^{2}\right]^{\theta},\alpha,\lambda,\beta,\theta>0,x\geq 0, \tag{7}$$

and

$$f(x;\alpha,\lambda,\beta,\theta)=2\alpha\beta\lambda\theta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1}\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta-1}\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}$$
$$\times\left[1-\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}^{2}\right]^{\theta-1},x>0. \quad (8)$$

If the random variable X follows (8), we will denoted by $X \sim \text{TLGIKw}(\alpha,\beta,\lambda,\theta)$. The hazard function $\tau(x)$ for the TLGIKw distribution is given by

$$\tau(x)=2\alpha\beta\lambda\theta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1}\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta-1}\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}$$
$$\times\left[1-\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}^{2}\right]^{\theta-1}\left\{1-\left[1-\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}^{2}\right]^{\theta}\right\}^{-1},x>0. \quad (9)$$

The plots of the density and hazard functions of the TLGIKw distribution are displayed in Fig. 1.

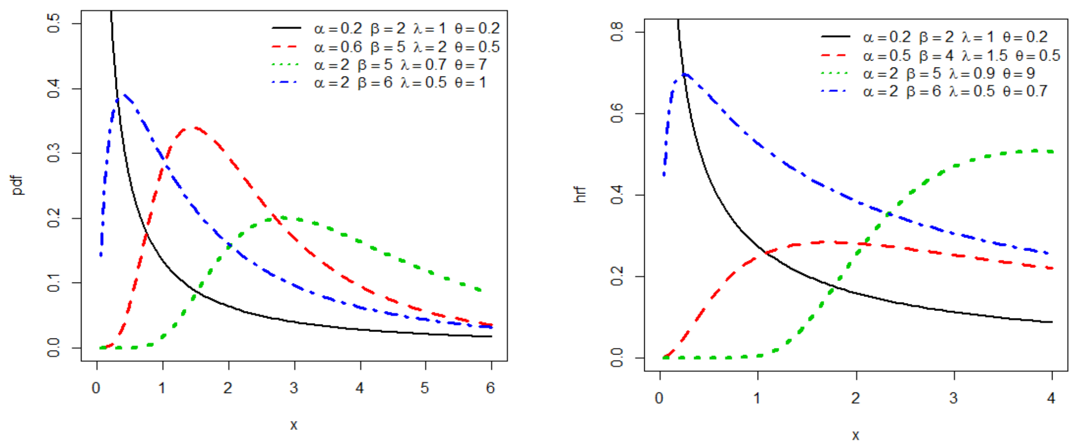

Fig. 1. Plots of the TLGIKw pdf and hrf for selected parameter values

The remainder of this paper is orgainzed as follows. In Section 2, an usefull expansion of the pdf corresponding to the new distribution is introduced. In Section 3, basic mathematical properties of the TLGIKw model are studied. In Section 4, the maximum likelihood estimates are obtained for the model parameters. A simulation study is conducted in Section 5. In Section 6, we provide two applications. Section 7 offers some concluding remarks.

2 Expansion of the Density Function

An usefull expansion of the density function of the TLGIKw distribution will be introduced in this section.

Applying the generalized binomial series in (8), we obtain

$$f(x;\phi)=2\alpha\beta\lambda\theta\sum_{j=0}^{\infty}\binom{\theta-1}{j}(-1)^{j}x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1}$$

$$\times\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta-1}\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}^{2j+1}$$

$$=2\alpha\beta\lambda\theta\sum_{j=0}^{\infty}\sum_{i=0}^{2j+1}\binom{\theta-1}{j}\binom{2j+1}{i}(-1)^{j+i}x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1}\left\{1-\left(1+x^{\lambda}\right)^{-\alpha}\right\}^{\beta(i+1)-1}$$

or

$$f(x)=\sum_{i=0}^{2j+1}\pi_i\, h_{\alpha,\beta(i+1),\lambda}(x), \tag{10}$$

where, $\pi_i=2\theta\sum_{j=0}^{\infty}(i+1)^{-1}(-1)^{j+i}\binom{\theta-1}{j}\binom{2j+1}{i}$ and $h_{\alpha,\beta(i+1),\lambda}(x)=\alpha\beta(i+1)\lambda x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1}\left\{1-\left(1+x^{\lambda}\right)^{-\alpha}\right\}^{\beta(i+1)-1}$ is the pdf of the GIKw distribution.

3 Mathematical Properties

In this section, we will investigate some main properties of the TLGIKw distribution.

3.1 Quantile function

The quantile function of the TLGIKw distribution $Q(u)=F^{-1}(u)$ for $u\in(0,1)$, $\alpha>0$, $\beta>0$, $\lambda>0$ and $\theta>0$ is the solution of the non-linear equation

$$Q(u)=G^{-1}\left\{\log\left\{\lambda^{-1}\left[\left\{1-\left[1-\left(1-u^{1/\theta}\right)^{1/2}\right]^{1/\beta}\right\}^{-1/\alpha}-1\right]\right\}\right\}. \tag{11}$$

3.2 Moments and generating functions

Let X be a random variable with the TLGIKw distribution, then the raw moment, say μ_r', is given by

$$\mu_r'=E(X^r)=\int_{-\infty}^{\infty}x^r f(x)\,dx$$

$$=\alpha\beta\lambda\sum_{i=0}^{2j+1}\pi_i\,(i+1)\int_0^{\infty}x^{r+\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1}\left\{1-\left(1+x^{\lambda}\right)^{-\alpha}\right\}^{\beta(i+1)-1}dx$$

Using the generalized binomial series in the above equation, we have

$$\mu_r'=\alpha\beta\lambda\sum_{i=0}^{2j+1}\sum_{\ell=0}^{\infty}\binom{\beta(i+1)-1}{\ell}(-1)^{\ell}\,\pi_i\,(i+1)\int_0^{\infty}x^{r+\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha(\ell+1)-1}dx$$

Letting $y=\left(1+x^{\lambda}\right)^{-1}$, we obtain

$$\mu_r' = \alpha\beta \sum_{i=0}^{2j+1}\sum_{\ell=0}^{\infty}\binom{\beta(i+1)-1}{\ell}(-1)^{\ell}\,\pi_i\,(i+1)\int_0^1 z^{\alpha(\ell+1)-r/\lambda-1}\left(1-z\right)^{r/\lambda}\,dz$$

$$= \sum_{i=0}^{2j+1}\sum_{\ell=0}^{\infty}\pi_i^{*}\,B\left(\alpha(\ell+1)-r/\lambda\,,r/\lambda+1\right), \tag{12}$$

where, $\pi_i^{*}=(-1)^{\ell}\alpha\beta(i+1)\binom{\beta(i+1)-1}{\ell}\pi_i$ and $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is the beta function.

Substituting $r=1,2,3,4$ in (12), we obtain the $\text{mean}=\mu_1'$, $\text{variance}=\mu_2'-\mu_1'^2$, $\text{skewness}=\mu_3'/\mu_1'^{3/2}$ and $\text{kurtosis}=\mu_4'/\mu_1'^2$.

The nth central moment of the TLGIKw distribution, say μ_n, can be obtained from

$$\mu_n = \sum_{r=0}^{n}\binom{n}{r}\left(-\mu_1'\right)^{n-r}E(x^r)$$

$$= \sum_{r=0}^{n}\sum_{i=0}^{2j+1}\sum_{\ell=0}^{\infty}\binom{n}{r}\left(-\mu_1'\right)^{n-r}\pi_i^{*}\,B\left(\alpha(\ell+1)-r/\lambda\,,r/\lambda+1\right). \tag{13}$$

The rth incomplete moment of the TLGIKw distribution, denoted by $\varphi_s(t)$, is

$$\varphi_s(t)=\int_{-\infty}^{t}x^{s}f(x)\,dx$$

$$=\alpha\beta\lambda\sum_{i=0}^{2j+1}\pi_i\,(i+1)\int_0^t x^{s+\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1}\left\{1-\left(1+x^{\lambda}\right)^{-\alpha}\right\}^{\beta(i+1)-1}dx$$

$$=\alpha\beta\lambda\sum_{i=0}^{2j+1}\sum_{\ell=0}^{\infty}\binom{\beta(i+1)-1}{\ell}(-1)^{\ell}\,\pi_i\,(i+1)\int_0^t x^{s+\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha(\ell+1)-1}dx$$

$$=\alpha\beta\lambda\sum_{i=0}^{2j+1}\sum_{\ell,w=0}^{\infty}(-1)^{\ell}\binom{\beta(i+1)-1}{\ell}\binom{-\alpha(\ell+1)-1}{w}\pi_i\,(i+1)\int_0^t x^{s+\lambda(w+1)-1}dx$$

$$=\sum_{i=0}^{2j+1}\sum_{\ell,w=0}^{\infty}\pi_i^{**}\,t^{s+\lambda(w+1)}, \tag{14}$$

where, $\pi_i^{**}=\dfrac{\lambda}{s+\lambda(w+1)}\binom{-\alpha(\ell+1)-1}{w}\pi_i^{*}$.

The moment and probability generating functions of the TLGIKw distribution, denoted by $M_X(t)$ and $M_{[X]}(t)$ respectively are given below:

$$M_X(t)=\sum_{i=0}^{2j+1}\sum_{r,\ell=0}^{\infty}\frac{t^r}{r!}\pi_i^*\,B\left(\alpha(\ell+1)-r/\lambda,r/\lambda+1\right). \quad (15)$$

and

$$M_{[X]}(t)=\sum_{i=0}^{2j+1}\sum_{r,\ell=0}^{\infty}\frac{\left(\ln t\right)^r}{r!}\pi_i^*\,B\left(\alpha(\ell+1)-r/\lambda,r/\lambda+1\right). \quad (16)$$

3.3 Probability weighted moments

The $(r+s)$th PWM of a random variable X with the TLGIKw distribution, say $M_{r,s}$, is given by

$$M_{r,s}=E\left(X^rF(x)^s\right)=\int_{-\infty}^{\infty}X^rF(x)^s f(x)dx. \quad (17)$$

From (7) and (8), we have

$$\begin{aligned}
f(x)F(x)^s &= 2\alpha\beta\lambda\theta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1}\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta-1} \\
&\times\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}\left[1-\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}^2\right]^{\theta(s+1)-1} \\
&=2\alpha\beta\lambda\theta\sum_{j=0}^{\infty}(-1)^j\binom{\theta(s+1)-1}{j}x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha-1} \\
&\times\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta-1}\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta}\right\}^{2j+1} \\
&=2\alpha\beta\lambda\theta\sum_{j=0}^{\infty}\sum_{i=0}^{2j+1}(-1)^{j+i}\binom{\theta(s+1)-1}{j}\binom{2j+1}{i}x^{\lambda-1} \\
&\times\left(1+x^{\lambda}\right)^{-\alpha-1}\left[1-\left(1+x^{\lambda}\right)^{-\alpha}\right]^{\beta(i+1)-1}
\end{aligned}$$

or

$$f(x)F(x)^s=\sum_{i=0}^{2j+1}\delta_i\,h_{\alpha,\beta(i+1),\lambda}(x), \quad (18)$$

where, $\delta_i=2\theta\sum_{j=0}^{\infty}(-1)^{j+i}\left(i+1\right)^{-1}\binom{\theta(s+1)-1}{j}\binom{2j+1}{i}.$

Subsituting from (18) in (17), we obtain

$$M_{r,s} == \sum_{i=0}^{2j+1}\sum_{\ell=0}^{\infty}\delta_i^{*}\,B\left(\alpha(\ell+1)-r/\lambda, r/\lambda+1\right), \tag{19}$$

where, $\delta_i^{*}=(-1)^{\ell}\alpha\beta(i+1)\binom{\beta(i+1)-1}{\ell}\delta_i.$

3.4 Order statistics

Let $X_{1:n} \le X_{2:n}, \ldots \le X_{n:n}$ be order statistics corresponding to a sample of size n from the TLGIKw distribution. The pdf of $X_{k:n}$, the kth order statistic, is given by

$$f_{X_{k:n}}(x)=\frac{1}{B(k,n-k+1)}\sum_{w=0}^{n-k}(-1)^{w}\binom{n-k}{w}f(x)F(x)^{k+w-1}. \tag{20}$$

Based on (7) and (8), we have

$$f(x)F(x)^{k+w-1}=\sum_{i=0}^{2j+1}\gamma_i\,h_{\alpha,\beta(i+1),\lambda}(x), \tag{21}$$

Where, $\gamma_i=2\theta\sum_{j=0}^{\infty}\left(i+1\right)^{-1}(-1)^{j+i}\binom{\theta(k+w)-1}{j}\binom{2j+1}{i}.$

Using (21) in (20), we arrive at

$$f_{X_{k:n}}(x)=\sum_{i=0}^{2j+1}\gamma_i^{*}\,h_{\alpha,\beta(i+1),\lambda}(x) \tag{22}$$

where, $\gamma_i^{*}=\sum_{w=0}^{n-k}\frac{(-1)^{w}}{B(k,n-k+1)}\binom{n-k}{w}\gamma_i.$

Furthermore, the rth moment of kth order statistic for the TLGIKw distribution is given by

$$E\left(x_{k:n}^{r}\right)=\sum_{i=0}^{2j+1}\sum_{\ell=0}^{\infty}\gamma_i^{**}\,B\left(\alpha(\ell+1)-r/\lambda, r/\lambda+1\right), \tag{23}$$

where, $\gamma_i^{**}=(-1)^{\ell}\alpha\beta\left(i+1\right)\binom{\beta(i+1)-1}{\ell}\gamma_i^{*}.$

3.5 Rényi entropy

The Rényi entropy is defined as

$$I_R(\delta)=\frac{1}{1-\delta}\left(\log I(\delta)\right), \tag{24}$$

where, $I(\delta)=\int f(x)^{\delta}\,dx,\ \delta>0$ and $\delta\neq 0.$

From (8), we have

$$I(\delta)=\sum_{\ell=0}^{\infty}\xi_{\ell}\,B\left(\alpha(\delta+\ell)+\frac{\delta-1}{\lambda},\delta-\frac{\delta-1}{\lambda}\right), \tag{25}$$

where, $\xi_{\ell}=\left(2\alpha\beta\theta\right)^{\delta}\lambda^{\delta-1}\sum_{j,i=0}^{\infty}(-1)^{j+i+\ell}\binom{\delta(\theta-1)}{j}\binom{\delta+2j}{i}\binom{\beta(\delta+i)-\delta}{\ell}.$

Inserting (25) in (24), we arrive at

$$I_{R}(\delta)=\frac{1}{1-\delta}\log\left\{\sum_{\ell=0}^{\infty}\xi_{\ell}\,B\left(\alpha(\delta+\ell)+\frac{\delta-1}{\lambda},\delta-\frac{\delta-1}{\lambda}\right)\right\}. \tag{26}$$

3.6 Stochastic ordering

Stochastic ordering is a vital criterion that is used in different fields to examine the comparative behavior. According to [8], a random variable X_1 is said to be smaller than another random variable X_2 in the likelihood ratio order $\left(X_1\leq_{lr}X_2\right)$ if $f_1(x)/f_2(x)$ decreases in x. The following theorem shows that the TLGIKw distribution is ordered in likelihood ratio ordering if the appropriate assumptions exist.

Theorem 1: Let $X_1\sim\text{TLGIKw}(\alpha_1,\beta_1,\lambda_1,\theta_1).$ and $X_2\sim\text{TLGIKw}(\alpha_2,\beta_2,\lambda_2,\theta_2).$ If $\beta_1=\beta_2,\quad \theta_1=\theta_2,\lambda_1\geq\lambda_2$ and $\alpha_1\geq\alpha_2,$ then $X_1\leq_{lr}X_2.$

Proof: We have

$$\frac{f_1(x)}{f_2(x)}=\left(\frac{\alpha_1\beta_1\lambda_1\theta_1}{\alpha_2\beta_2\lambda_2\theta_2}\right)x_i^{\lambda_1-\lambda_2}\,\mu_i^{-(\alpha_1+1)}\sigma_i^{\alpha_2+1}\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1-1}\left(1-\sigma_i^{-\alpha_2}\right)^{-(\beta_2-1)}$$
$$\times\left[1-\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1}\right]\left[1-\left(1-\sigma_i^{-\alpha_2}\right)^{\beta_2}\right]^{-1}\left\{1-\left[1-\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1}\right]^{2}\right\}^{\theta_1-1}$$
$$\times\left\{1-\left[1-\left(1-\sigma_i^{-\alpha_2}\right)^{\beta_2}\right]^{2}\right\}^{-(\theta_2-1)}$$

where, $\mu_i=1+x_i^{\lambda_1}$ and $\sigma_i=1+x_i^{\lambda_2}.$

Then

$$\log\frac{f_1(x)}{f_2(x)}=\log\left(\frac{\alpha_1\beta_1\lambda_1\theta_1}{\alpha_2\beta_2\lambda_2\theta_2}\right)+\left(\lambda_1-\lambda_2\right)\log(x_i)-(\alpha_1+1)\log\left(\mu_i\right)$$
$$+(\alpha_2+1)\log\left(\sigma_i\right)+(\beta_1-1)\log\left(1-\mu_i^{-\alpha_1}\right)-(\beta_2-1)\log\left(1-\sigma_i^{-\alpha_2}\right)$$

$$+\log\left[1-\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1}\right]-\log\left[1-\left(1-\sigma_i^{-\alpha_2}\right)^{\beta_2}\right]$$

$$+(\theta_1-1)\log\left\{1-\left[1-\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1}\right]^2\right\}-(\theta_2-1)\log\left\{1-\left[1-\left(1-\sigma_i^{-\alpha_2}\right)^{\beta_2}\right]^2\right\}$$

If $\beta_1=\beta_2,\ \theta_1=\theta_2,\ \lambda_1\geq\lambda_2$ and $\alpha_1\geq\alpha_2$, then we have

$$\frac{d}{dx}\log\frac{f_1(x)}{f_2(x)}=\left(\lambda_1-\lambda_2\right)x_i^{-1}-\lambda_1(\alpha_1+1)x_i^{\lambda_1-1}\mu_i^{-1}+\lambda_2(\alpha_2+1)x_i^{\lambda_2-1}\sigma_i^{-1}$$

$$+\alpha_1\lambda_1(\beta_1-1)x_i^{\lambda_1-1}\mu_i^{-(\alpha_1+1)}\left(1-\mu_i^{-\alpha_1}\right)^{-1}$$

$$-\alpha_2\lambda_2(\beta_2-1)x_i^{\lambda_2-1}\sigma_i^{-(\alpha_2+1)}\left(1-\sigma_i^{-\alpha_2}\right)^{-1}$$

$$-\alpha_1\beta_1\lambda_1x_i^{\lambda_1-1}\mu_i^{-(\alpha_1+1)}\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1-1}\left\{1-\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1}\right\}^{-1}$$

$$+\alpha_2\beta_2\lambda_2x_i^{\lambda_2-1}\sigma_i^{-(\alpha_2+1)}\left(1-\sigma_i^{-\alpha_2}\right)^{\beta_2-1}\left\{1-\left(1-\sigma_i^{-\alpha_2}\right)^{\beta_2}\right\}^{-1}$$

$$+2\alpha_1\beta_1\lambda_1\left(\theta_1-1\right)x_i^{\lambda_1-1}\mu_i^{-(\alpha_1+1)}\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1-1}$$

$$\times\left[1-\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1}\right]\left\{1-\left[1-\left(1-\mu_i^{-\alpha_1}\right)^{\beta_1}\right]^2\right\}^{-1}$$

$$-2\alpha_2\beta_2\lambda_2\left(\theta_2-1\right)x_i^{\lambda_2-1}\sigma_i^{-(\alpha_2+1)}\left(1-\sigma_i^{-\alpha_2}\right)^{\beta_2-1}$$

$$\times\left[1-\left(1-\sigma_i^{-\alpha_2}\right)^{\beta_2}\right]\left\{1-\left[1-\left(1-\sigma_i^{-\alpha_2}\right)^{\beta_2}\right]^2\right\}^{-1}<0$$

Consequentely, $f_1(x)/f_2(x)$ decreases in x and hence $X_1\leq_{lr}X_2$.

3.7 Stress strength model

Let X_1 and X_2 be two independent random variables with $X_1\sim\text{TLGIKw}(\alpha_1,\beta_1,\lambda_1,\theta_1)$ and $X_2\sim\text{TLGIKw}(\alpha_2,\beta_2,\lambda_2,\theta_2)$ distributions. Then, the stress strength model is given by

$$R=\Pr(X_2<X_1)=\int_0^\infty f_1\left(\alpha_1,\beta_1,\lambda_1,\theta_1\right)F_2\left(\alpha_2,\beta_2,\lambda_2,\theta_2\right)dx \tag{27}$$

Using (7) and (8), we obtain

$$f_1\left(\alpha_1,\beta_1,\lambda_1,\theta_1\right)F_2\left(\alpha_2,\beta_2,\lambda_2,\theta_2\right)=\sum_{q,s=0}^{\infty}\Omega_{q,s}\,x^{\lambda_1+\lambda_2 s}\left(1+x^{\lambda_1}\right)^{-\alpha_1(q+1)-1}, \tag{28}$$

where, $\Omega_{q,s}=2\alpha_1\beta_1\lambda_1\theta_1\sum_{j,w,m=0}^{\infty}\sum_{i=0}^{2j+1}\sum_{\ell=0}^{2w}(-1)^{j+w+i+\ell+q+m}\binom{\theta_1-1}{j}\binom{\theta_2}{w}\binom{2j+1}{i}\binom{2w}{\ell}\binom{\beta_1(i+1)-1}{q}\binom{\beta_2\ell}{m}\binom{-\alpha_2 m}{s}.$

Subsituting from (28) in (27), we have

$$R=\sum_{q,s=0}^{\infty}\Omega_{q,s}\int_0^{\infty}x^{\lambda_1+\lambda_2 s}\left(1+x^{\lambda_1}\right)^{-\alpha_1(q+1)-1}dx$$

$$=\lambda_1^{-1}\sum_{q,s=0}^{\infty}\Omega_{q,s}\int_0^{\infty}z^{\left(\alpha_1(q+1)+(1+\lambda_2 s)/\lambda_1\right)-1}\left(1-z\right)^{(\lambda_2 s-1)/\lambda_1}dx$$

Or

$$R=\lambda_1^{-1}\sum_{q,s=0}^{\infty}\Omega_{q,s}\,B\left(\alpha_1(q+1)+(1+\lambda_2 s)/\lambda_1,\left((\lambda_2 s-1)/\lambda_1\right)+1\right). \tag{29}$$

4 Maximum Likelihood Estimation

The maximum likelihood estimates (MLEs) for the model parameters of the TLGIKw distribution will be discussed in this section. Let $x_1,x_2,...,x_n$ be observed values of a random sample from the TLGIKw distribution, then the corresponding log-likelihood function is given by

$$\ell=n\left\{\log(2)+\log(\alpha)+\log(\beta)+\log(\lambda)+\log(\theta)\right\}+(\lambda-1)\sum_{i=1}^{n}\log(x_i)$$
$$-(\alpha+1)\sum_{i=1}^{n}\log(a_i)+(\beta-1)\sum_{i=1}^{n}\log\left(1-a_i^{-\alpha}\right)+\sum_{i=1}^{n}\log\left\{1-\left(1-a_i^{-\alpha}\right)^{\beta}\right\}$$
$$+(\theta+1)\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1-a_i^{-\alpha}\right)^{\beta}\right]^2\right\}, \tag{30}$$

where, $a_i=1+x_i^{\lambda}$.

By differentiating (30) with respect to α,β,λ and $\theta,$ we obtain the following nonlinear system of equations

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\log(a_i)+(\beta-1)\sum_{i=1}^{n}\left\{\frac{a_i^{-\alpha}\log(a_i)}{1-a_i^{-\alpha}}\right\}-\beta\sum_{i=1}^{n}\left\{\frac{a_i^{-\alpha}\left(1-a_i^{-\alpha}\right)^{\beta-1}\log(a_i)}{1-\left(1-a_i^{-\alpha}\right)^{\beta}}\right\}$$
$$+2\beta(\theta-1)\sum_{i=1}^{n}\left\{\frac{a_i^{-\alpha}\left(1-a_i^{-\alpha}\right)^{\beta-1}\left[1-\left(1-a_i^{-\alpha}\right)^{\beta}\right]\log(a_i)}{1-\left[1-\left(1-a_i^{-\alpha}\right)^{\beta}\right]^2}\right\}, \tag{31}$$

$$\frac{\partial \ell}{\partial \beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\log\left(1-a_i^{-\alpha}\right)-\sum_{i=1}^{n}\left\{\frac{\left(1-a_i^{-\alpha}\right)^{\beta}\log\left(1-a_i^{-\alpha}\right)}{1-\left(1-a_i^{-\alpha}\right)^{\beta}}\right\}$$
$$-2\left(\theta-1\right)\sum_{i=1}^{n}\left\{\frac{\left(1-a_i^{-\alpha}\right)^{\beta}\left[1-\left(1-a_i^{-\alpha}\right)^{\beta}\right]\log\left(1-a_i^{-\alpha}\right)}{1-\left[1-\left(1-a_i^{-\alpha}\right)^{\beta}\right]^{2}}\right\}, \tag{32}$$

$$\frac{\partial \ell}{\partial \lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log\left(x_i\right)-\left(\alpha+1\right)\sum_{i=1}^{n}\left\{\frac{x_i^{\lambda}\log\left(x_i\right)}{a_i}\right\}+\alpha\left(\beta-1\right)\sum_{i=1}^{n}\left\{\frac{x_i^{\lambda}\,a_i^{-\alpha-1}\log\left(x_i\right)}{1-a_i^{-\alpha}}\right\}$$
$$-\alpha\beta\sum_{i=1}^{n}\left\{\frac{x_i^{\lambda}\,a_i^{-\alpha-1}\left(1-a_i^{-\alpha}\right)^{\beta-1}\log\left(x_i\right)}{1-\left(1-a_i^{-\alpha}\right)^{\beta}}\right\}$$
$$+2\alpha\beta\left(\theta-1\right)\sum_{i=1}^{n}\left\{\frac{x_i^{\lambda}\,a_i^{-\alpha-1}\left(1-a_i^{-\alpha}\right)^{\beta-1}\left[1-\left(1-a_i^{-\alpha}\right)^{\beta}\right]\log\left(x_i\right)}{1-\left[1-\left(1-a_i^{-\alpha}\right)^{\beta}\right]^{2}}\right\} \tag{33}$$

$$\frac{\partial \ell}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1-a_i^{-\alpha}\right)^{\beta}\right]^{2}\right\}. \tag{34}$$

The MLEs, say $\hat{\Theta}=(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})$, of $\Theta=(\alpha,\beta,\lambda,\theta)^T$ can be obtained by equating the system of nonlinear equations (31) to (34) to zero and solving them simultaneously by using statistical software.

5 Simulation Study

It is very difficult to compare the theoretical performances of the different estimators (MLEs) for the TLGIKw distribution. Therefore, simulation is needed to compare the performances of the different methods of estimation mainly with respect to their biases, mean square errors and Variances (MLEs) for different sample sizes. A numerical study is performed using Mathematica 9 software. Different sample sizes are considered through the experiments at size sizes $n = 50,100,200,300$ and 500. In addition, the different values of parameters $(\alpha,\beta,\lambda,\theta)$.

The experiment will be repeated 3000 times. In each experiment, the estimates of the parameters will be obtained by maximum likelihood methods of estimation. The means, MSEs and biases for the different estimators will be reported from these experiments.

We can observe from Table 1 that, if the sample size increases, the empirical biases and MSEs decreases in all cases.

6 Applications

This section represents the potentiality and flexibility of the new model as compared to some other existing life-time models by using some real-life examples. The first application is initially used by [9]. The observed values are: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2. The second real-life data was originally reported by [10]. The data consists of 30 observations of the March precipitation (in inches) in Minneapolis/St Paul. The observed values are: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Table 1. The parameter estimation from the TLGIKw distribution using MLEs

n	Parameters	Initial Values	MLEs	Bias	MSE
50	α	0.7	0.71520	0.01920	0.01602
	β	0.5	0.52758	0.02868	0.05168
	λ	0.5	0.52605	0.03605	0.02884
	θ	1.5	1.55390	0.08322	0.15907
100	α	0.7	0.71189	0.01289	0.00762
	β	0.5	0.52033	0.02093	0.02444
	λ	0.5	0.51629	0.01329	0.00860
	θ	1.5	1.55347	0.05231	0.08030
200	α	0.7	0.70767	0.00751	0.00595
	β	0.5	0.51242	0.01231	0.01779
	λ	0.5	0.51112	0.01102	0.00539
	θ	1.5	1.53394	0.03294	0.05551
300	α	0.7	0.70409	0.00419	0.00304
	β	0.5	0.51001	0.00649	0.09390
	λ	0.5	0.50207	0.00551	0.00501
	θ	1.5	1.51039	0.01005	0.00255
500	α	0.7	0.70298	0.00222	0.00235
	β	0.5	0.50380	0.00352	0.00593
	λ	0.5	0.50445	0.00436	0.00205
	θ	1.5	1.50836	0.00620	0.02012

Table 2. MLEs estimates of the parameters with the goodness of fit measures for data set 1.

Models	Estimates (Std. Error)				K-S	A*	W*
TLGIKw	1.8917	19.0536	0.7962	0.1473	0.080	0.208	0.028
$(\alpha,\beta,\lambda,\theta)$	(0.1685)	(0.2619)	(0.1135)	(0.0344)			
MOEIK	2.1193	1.7026	2.2471	---	0.082	0.233	0.035
(α,β,λ)	(0.5717)	(0.8020)	(2.3570)				
GIK	2.0236	2.6369	0.8858	---	0.095	0.281	0.039
(α,β,γ)	(1.8839)	(3.8684)	(0.6642)				
TEIR	6.9784	0.0138	0.7798	---	0.429	18.01	2.294
(α,θ,λ)	(12.604)	(0.0247)	(0.1350)				
LW	2.7709	0.7056	0.5526	---	0.085	6.327	0.633
(α,β,λ)	(193.53)	(17.189)	(38.601)				
TPL	0.9265	0.7453	0.4094	---	0.091	0.280	0.048
(α,θ,λ)	(0.1156)	(0.2414)	(0.5958)				
MOFr	29.053	1.4730	0.1124	---	0.089	0.289	0.043
(α,θ,σ)	(2.3853)	(0.2397)	(0.1098)				
IK	1.7409	2.1058	---	---	0.096	0.279	0.039
(α,β)	(0.3238)	(0.5373)					

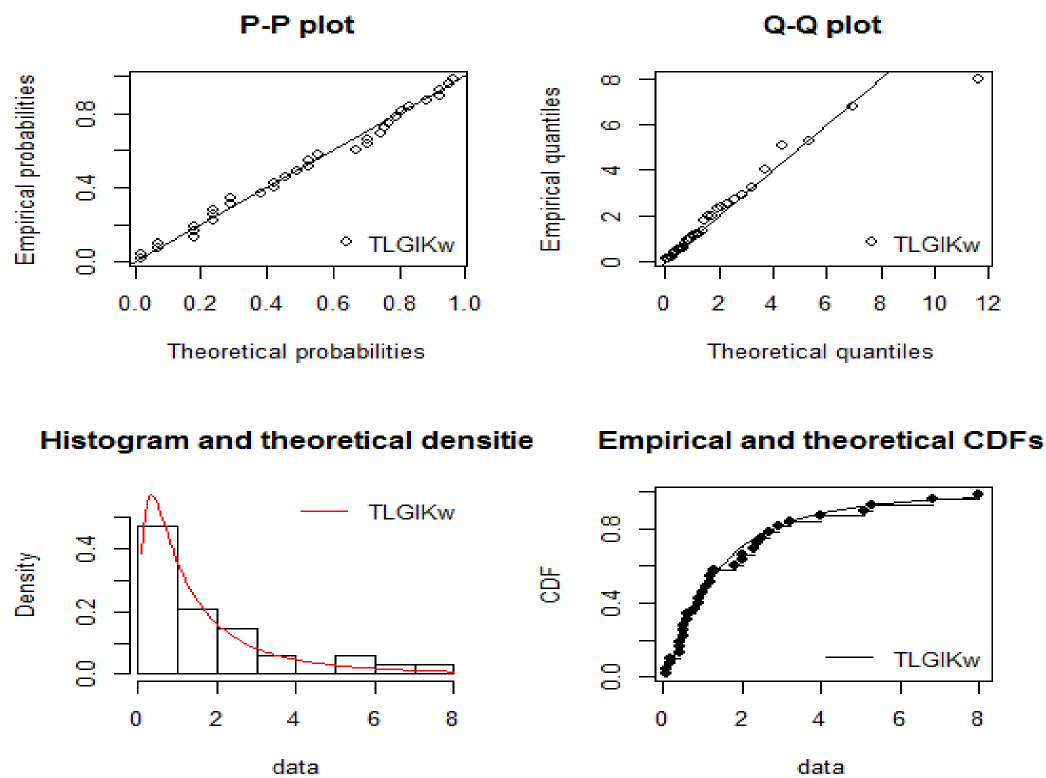

Fig. 2. PP, QQ, epdf and ecdf plots of the TLGIKw distribution for data set 1.

Table 3. MLEs estimates of the parameters with the goodness of fit measures for data set 2.

Models	Estimates (Std. error)				K-S	A*	W*
TLGIKw ($\alpha,\beta,\lambda,\theta$)	1.8972 (1.6060)	20.9356 (3.5601)	1.2356 (0.9897)	0.2082 (0.3095)	0.043	0.114	0.017
MOEIK (α,β,λ)	4.3228 (0.9387)	6.5798 (5.0017)	6.9226 (9.6043)	---	0.068	0.137	0.019
GIK (α,β,γ)	1.9552 (1.8117)	3.9501 (5.6951)	1.4202 (1.0211)	---	0.110	0.318	0.052
TEIR (α,θ,λ)	6.5630 (80.152)	0.0958 (1.1693)	0.6700 (0.2661)	---	0.182	1.139	0.212
LW (α,β,λ)	2.7709 (243.47)	0.3635 (32.322)	1.0061 (88.405)	---	0.074	0.191	0.025
TPL (α,θ,λ)	1.5965 (0.2054)	0.4801 (0.1643)	0.5812 (0.6548)	---	0.091	0.146	0.029
MOFr (α,θ,σ)	42.598 (2.8490)	2.6975 (0.4482)	0.3548 (0.2109)	---	0.095	0.224	0.024
IK (α,β)	2.9872 (0.4730)	8.5899 (3.1222)	---	---	0.114	0.338	0.055

By using these data sets, we have made comparison for the new TLGIKw with generated Marshall-Olkin extended inverted Kumaraswamy (MOEIK) distribution by [11], the Generalized Inverted Kumaraswamy (GIK) by [6], Transmuted Exponentiated Inverse Rayleigh (TEIR) by [12], Logistic Weibull (LW) by [13], Transmuted Power Lindley (TPL) by [14], Marshall Olkin Frechet (MOFr) by [15] and Inverted Kumaraswamy (IK) distribution, by [5]. We use numerous goodness of fit measures to compare the new developed model with other existing models such as Kolmogrov Smirnov (K-S), Anderson Darling (A*) and Cramer-von Mises(W*). Tables (2) and (3) provide the values of goodness-of-fit measures for the TLGIKw model and other fitted models, with the MLEs and their corresponding standard errors (SEs) (in parentheses) for the two data sets respectively. The plots of the fitted model for the two data sets are shown respectively in Figs. 2 and 3.

The values in Tables (2), (3) and the plots in Figs. (2), (3) indicate that the TLGIKW distribution yields the best fit among all compatative models.

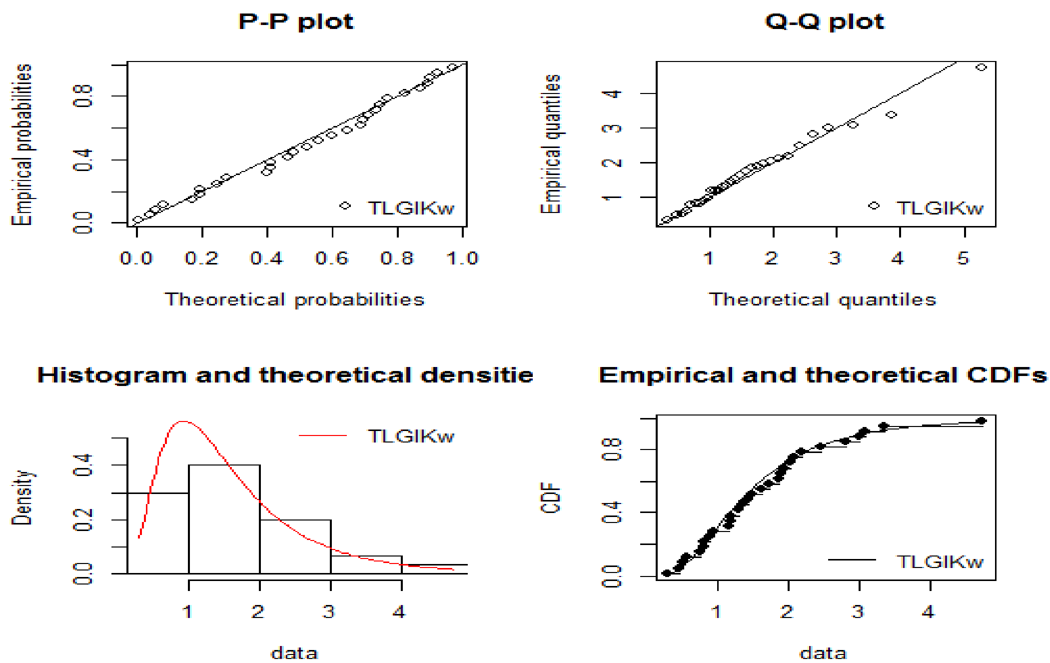

Fig. 3. PP, QQ, epdf and ecdf plots of the TLGIKw distribution for data set 2.

7 Conclusions

We introduce a new member of the inverted distributions called the Topp Leone generalized inverted Kumaraswamy distribution. The main features of the new model such as the quantile function, ordinary and incomplete moments, generating functions, order statistics, Rényi entropy, stress strength model and stochastic ordering are derived. The maximum likelihood estimates are obtaine for the model parameters and the importance of these estimates are assesed by means of a simulation study. The usefulness of the new model is illustrated via two real applications. Numerical results show that the new distribution can be considered a good alternative to some well known distributions.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] AL-Dayian GR. Burr Type III distribution: Properties and estimation. The Egyptian Statistical Journal. 1999;43:102-116.

[2] Abd EL-Kader RI. A general class of some inverted distributions. Ph.D. Thesis, AL-Azhar University, Girls' Branch, Egypt, Cairo; 2013.

[3] Prakash G. Inverted exponential distribution under a Bayesian view point. Journal of Modern Applied Statistical Methods. 2012;11(1):190-202.

[4] Aljuaid A. Estimating the parameters of an exponentiated inverted Weibull distribution under Type II censoring. Applied Mathematical Sciences. 2013;7(35):1721-1736.

[5] Abd AL-Fattah, A.M., EL-Helbawy, A.A. and AL-Dayian, G.R. Inverted Kumaraswamy distribution: properties and estimation. Pak. J. Statist., 2017, 33(1): 37-61.

[6] Iqbal Z, Tahir MM, Azeem S, Ahmad M. Generalized inverted Kumaraswamy distribution: Properties and Application. Open Journal of Statistics. 2017;7:645-662.

[7] Al-Shomrani A, Arif O, Shawky K, Hanif S, Shahbaz MQ. Topp Leone family of distributions: Some properties and application. Pak. J. Stat. Oper. Res. 2016;443-451.

[8] Shaked M, Shanthikumar JG. Stochastic Orders. Wiley: New York; 2007.

[9] Bhaumik DK, Kapur K, Gibbons RD. Testing parameters of a gamma distribution for small samples. Technometrics. 2009;51(3):326-334.

[10] Hinkley D. On quick choice of power transformation. Appl. Stat. 1977;67-69.

[11] Usman RM, Muhammad Ahsan ul Haq. The Marshall-Olkin extended inverted Kumaraswamy distribution: theory and applications. Journal of King Saud University-Science. 2018.
Available:https://doi.org/10.1016/j.jksus.2018.05.021

[12] Muhammad Ahsan ul Haq. Transmuted exponentiated inverse Rayleigh distribution. J. Stat. Appl. Prob. 2016;5(2):337-343.

[13] Tahir M, Cordeiro GM, Alzaatreh A, Mansoor M, Zubair M. The Logistic-X family of distributions and its applications. Commun. Stat. Theory Methods. 2016;45(24):7326-7349.

[14] Granzotto DCT, Mazucheli J, Louzada F. Statistical study of monthly rainfall trends by using the transmuted power Lindley distribution. Int. J. Stat. Prob. 2016;6(1):111-125.

[15] Krishna E, Jose KK, Ristic MM. Applications of Marshall-Olkin Fréchet distribution. Commun. Stat. Simul. Comput. 2013;42(1):76–89.

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