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On Some Generalized Fixed Point Theorems in 2 – Metric Space

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Authors' contributions

This work was carried out in collaboration between all authors. Author VSK contributed theorems and proofs in section author BRR tested the validity of the proofs and made corrections. Author TVLN contributed preliminaries and worked together with author VSK. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, some fixed point theorems in 2-metric spaces are established. Some of the theorems are generalizations of fixed point theorems that were proved by *Lal* and *Singh* in 2-metric spaces.

Keywords: Fixed point; 2 – metric space; complete 2-metric space; contraction type mapping.

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1. INTRODUCTION

In 1906, Frechet [1,2] introduced the notion of metric as an abstract generalization of length concept. In 1963, Gahler [3] introduced the notion of 2-metric as an abstract generalization of the notion of area function for Euclidean triangles. Many fixed point theorems appeared in 2-metric spaces in later years analogous to the fixed point theorems in metric spaces proved by various authors like Iseki [4], Lal and Singh [5], Rhoades [6] etc.

In this present work, we generalize the fixed point theorems that are proved by Lal and Singh [5]. In what follows X and $\boldsymbol{R}$ stand for a non-empty set and the real line respectively.

1.1 Preliminaries

In this section, we present some basic definitions which are needed for the further study of this paper.

1.1.1 Definition

A point $x \in X$ is said to be a *fixed point* of a self-map $f : X \to X$ if $f(x) = x$.

1.1.2 Definition

Let X be a non-empty set and $d : X \times X \times X \to \boldsymbol{R}$. For all x, y, z and u in X, if d satisfies the following conditions

(a) $d(x, y, z) = 0$ if at least two of x, y, z are equal
(b) $d(x, y, z) = d(x, z, y) = d(y, z, x) = ...$
(c) $d(x,y,z) \le d(x,y,u) + d(x,u,z) + d(u,y,z)$

Then d is called a *2-metric* on X and the pair (X, d) is called a *2-metric space.*

1.1.3 Definition

Let (X, d) be a 2-metric space. A sequence $\{x_n\}$ in X is called a *Cauchy sequence,* if $d(x_m, x_n, a) \to 0$ as $m, n \to \infty$ for all $a \in X$.

1.1.4 Definition

Let (X, d) be a 2-metric space. A sequence $\{x_n\}$ is said to *converge* to a point x in X if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for every a in X.

1.1.5 Definition

A 2-metric space (X, d) is said to be a *complete 2-metric space* if every Cauchy sequence in X *converges* in X.

1.1.6 Remark

Let $\{x_n\}$ be a sequence in a complete 2-metric space (X, d). If there exists $r \in (0,1)$ such that $d(x_n, x_{n+1}, a) \le r^n\, d(x_{n-1}, x_n, a)$ for every a in X and every non-negative integer n, then $\{x_n\}$ converges to a point in X.

1.1.7 Definition

Let (X, d) be a 2-metric space. A mapping $T : X \to X$ is called a contraction on X if there exists a real number h with $0 < h < 1$ such that $d(Tx, Ty, a) \le hd(x, y, a)$ for all x, y, a in X.

1.2 Some Fixed Point Theorems for Self - maps

Several theorems have been proved for the existence of fixed points in 2-metric spaces. In 1978, Lal and singh [5] proved some fixed point theorems for self-maps on a complete 2-metric space. In this section, we generalize those fixed point theorems.

1.2.1 Theorem

Let $0 < \alpha < 1$. Let p and q be non-negative real numbers such that $p + q < 1$. Suppose that f and g are two self maps on a complete 2-metric space (X, d). If

(a) $\alpha|p-q| \leq 1-(p+q)$

(b) $d\left(f(x), g(y), a\right) \leq \alpha \max\left\{d(x,y,a), d(x,f(x),a), d(y,g(y),a)\right\} + (1-\alpha)\left(p\, d\left(x,g(y),a\right) + q d\left(y,f(x),a\right)\right)$

whenever x, y, a are distinct points in X, then f and g have a unique common fixed point in X.

Proof: Fix $x_0 \in X$. Define a sequence $\{x_n\}$ in X as follows.

$$x_{2n+1} = f\left(x_{2n}\right)$$

$$x_{2n+2} = g(x_{2n+1}) \text{ for } n = 0,1,2,...$$

First we show that $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$. We have

$$\begin{aligned} d(x_{2n}, x_{2n+1}, x_{2n+2}) &= d(x_{2n+1}, x_{2n+2}, x_{2n}) \\ &= d(f(x_{2n}), g(x_{2n+1}), x_{2n}) \\ &\leq \alpha \max\left\{d\left(x_{2n}, x_{2n+1}, x_{2n}\right), d\left(x_{2n}, f(x_{2n}), x_{2n}\right),\right. \\ &\quad \left. d\left(x_{2n+1}, g(x_{2n+1}), x_{2n}\right)\right\} + (1-\alpha)\left[p\, d\left(x_{2n}, g\left(x_{2n+1}\right), x_{2n}\right)\right. \\ &\quad \left. + q\, d\left(x_{2n+1}, f\left(x_{2n}\right), x_{2n}\right)\right] \\ &= \alpha \max\left\{d\left(x_{2n}, x_{2n+1}, x_{2n}\right), d\left(x_{2n}, x_{2n+1}, x_{2n}\right), d\left(x_{2n+1}, x_{2n+2}, x_{2n}\right)\right\} \\ &\quad + (1-\alpha)\left[p\, d\left(x_{2n}, x_{2n+2}, x_{2n}\right) + q\, d\left(x_{2n+1}, x_{2n+1}, x_{2n}\right)\right] \\ &= \alpha\, d\left(x_{2n+1}, x_{2n+2}, x_{2n}\right) \end{aligned}$$

$$\Rightarrow d\left(x_{2n}, x_{2n+1}, x_{2n+2}\right) \leq \alpha\, d\left(x_{2n}, x_{2n+1}, x_{2n+2}\right) < d\left(x_{2n}, x_{2n+1}, x_{2n+2}\right) \text{ (since } 0 < \alpha < 1\text{)}$$

$$\Rightarrow d\left(x_{2n}, x_{2n+1}, x_{2n+2}\right) = 0 \qquad \rightarrow (1)$$

Now we have

$$\begin{aligned} d\left(x_{2n+1}, x_{2n+2}, a\right) &= d\left(f\left(x_{2n}\right), g\left(x_{2n+1}\right), a\right) \\ &\leq \alpha \max\left\{d\left(x_{2n}, x_{2n+1}, a\right), d\left(x_{2n}, f\left(x_{2n}\right), a\right), d\left(x_{2n+1}, g\left(x_{2n+1}\right), a\right)\right\} \\ &\quad + (1-\alpha)\left[p\, d\left(x_{2n}, g\left(x_{2n+1}\right), a\right) + q\, d\left(x_{2n+1}, f\left(x_{2n}\right), a\right)\right] \\ &= \alpha \max\left\{d\left(x_{2n}, x_{2n+1}, a\right), d\left(x_{2n+1}, x_{2n+2}, a\right)\right\} + (1-\alpha) \\ &\quad pd\left(x_{2n}, x_{2n+2}, a\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow d\left(x_{2n+1}, x_{2n+2}, a\right) &\leq \alpha \max\left\{d(x_{2n}, x_{2n+1}, a), d(x_{2n+1}, x_{2n+2}, a)\right\} + (1-\alpha) \\ &\quad p\, d\left(x_{2n}, x_{2n+2}, a\right) \end{aligned} \qquad \rightarrow (2)$$

Put $M = \max\left\{d(x_{2n}, x_{2n+1}, a), d(x_{2n+1}, x_{2n+2}, a)\right\}$. Then the following cases are arise.

Case – 1

Suppose that $M = d\left(x_{2n}, x_{2n+1}, a\right)$.

Then $d\left(x_{2n+1},x_{2n+2},a\right) \le \alpha\, d\left(x_{2n},x_{2n+1},a\right) + (1-\alpha)\, p\, d\left(x_{2n},x_{2n+2},a\right)$

$$\le \alpha\, d\left(x_{2n},x_{2n+1},a\right) + (1-\alpha)\, p\left(d\left(x_{2n},x_{2n+2},x_{2n+1}\right) + d\left(x_{2n},x_{2n+1},a\right) + d\left(x_{2n+1},x_{2n+2},a\right)\right)$$

$$= \alpha\, d\left(x_{2n},x_{2n+1},a\right) + (1-\alpha)\, p\left(d\left(x_{2n},x_{2n+1},a\right) + d\left(x_{2n+1},x_{2n+2},a\right)\right) \text{ (since } d\left(x_{2n},x_{2n+2},x_{2n+1}\right)=0\text{)}$$

$$\Rightarrow\ d\left(x_{2n+1},x_{2n+2},a\right) \le \frac{\alpha+(1-\alpha)p}{1-(1-\alpha)p}\, d(x_{2n},x_{2n+1},a)$$

Put $\beta = \dfrac{\alpha+(1-\alpha)p}{1-(1-\alpha)p}$

From the inequality (a), it follows that $0<\beta<1$.

Then $d\left(x_{2n+1},x_{2n+2},a\right) \le \beta\, d\left(x_{2n},x_{2n+1},a\right)$ $\rightarrow (3)$

Case – 2

Suppose that $M = d\left(x_{2n+1},x_{2n+2},a\right)$.

Then $d\left(x_{2n+1},x_{2n+2},a\right) \le \alpha\, d\left(x_{2n+1},x_{2n+2},a\right) + (1-\alpha)\, p\, d\left(x_{2n},x_{2n+2},a\right)$

$$\Rightarrow\ (1-\alpha)\, d\left(x_{2n+1},x_{2n+2},a\right) \le (1-\alpha)\, p\, d\left(x_{2n},x_{2n+2},a\right)$$

$$\Rightarrow\ d\left(x_{2n+1},x_{2n+2},a\right) \le p\, d\left(x_{2n},x_{2n+2},a\right)$$

$$\le p\, d\left(x_{2n},x_{2n+1},a\right) + q\, d\left(x_{2n+1},x_{2n+2},a\right)$$

$$\Rightarrow\ (1-q)\, d\left(x_{2n+1},x_{2n+2},a\right) \le p\, d\left(x_{2n},x_{2n+1},a\right)$$

$$\Rightarrow\ d\left(x_{2n+1},x_{2n+2},a\right) \le \left(\frac{p}{1-q}\right) d\left(x_{2n},x_{2n+1},a\right)$$

Put $\gamma = \left(\dfrac{p}{1-q}\right)$

$$d\left(x_{2n+1},x_{2n+2},a\right) \le \gamma\, d\left(x_{2n},x_{2n+1},a\right) \qquad \rightarrow (4)$$

Since $p+q<1$, $p<1-q \quad \Rightarrow \quad \dfrac{p}{1-q}<1 \quad \Rightarrow \quad 0<\gamma<1$

From (3) and (4) we have

$$d\left(x_{2n+1},x_{2n+2},a\right) \le \max\{\beta,\gamma\}\, d\left(x_{2n},x_{2n+1},a\right) \qquad \rightarrow (5)$$

Similarly

$$d\left(x_{2n},x_{2n+1},a\right) = d\left(g(x_{2n-1}),f(x_{2n}),a\right)$$

$$= d\left(f(x_{2n}),g(x_{2n-1}),a\right)$$

$$\le \max\{\beta,\gamma\}\, d\left(x_{2n-1},x_{2n},a\right) \qquad \rightarrow (6)$$

Put $\delta = \max\{\beta,\gamma\}$ In (5) and (6). Then we get the following

$$d\left(x_{2n+1}, x_{2n+2}, a\right) \leq \delta\, d\left(x_{2n}, x_{2n+1}, a\right)$$
$$d\left(x_{2n}, x_{2n+1}, a\right) \leq \delta\, d\left(x_{2n-1}, x_{2n}, a\right)$$

Similarly $d\left(x_{2n-1}, x_{2n}, a\right) \leq \delta\ d\left(x_{2n-2}, x_{2n-1}, a\right)$

From these three, we get

$$d\left(x_{2n+1}, x_{2n+2}, a\right) \leq \delta^3 d\left(x_{2n-2}, x_{2n-1}, a\right)$$

Proceeding in this way, we obtain

$d\left(x_{2n+1}, x_{2n+2}, a\right) \leq \delta^n d(x_0, x_1, a)$ where $0 < \delta < 1$

$\Rightarrow d\left(x_{2n+1}, x_{2n+2}, a\right) \to 0$ as $n \to \infty$ for every a in X.

$\Rightarrow \{x_n\}$ is a Cauchy Sequence in X .

Since X is complete 2-metric space, $\{x_n\}$ converges in X.

$\Rightarrow$ there exists a point $z \in X$ such that $\lim_{n\to\infty} x_n = z$ in X.

Now we prove that z is a common fixed point of both f and g in X. By the properties of 2-metric, we have

$$\begin{aligned}
d\left(z, g(z), a\right) &\leq d\left(z, g(z), x_{2n+1}\right) + d\left(z, x_{2n+1}, a\right) + d\left(x_{2n+1}, g(z), a\right) \\
&= d\left(z, g(z), x_{2n+1}\right) + d\left(z, x_{2n+1}, a\right) + d\left(f(x_{2n}), g(z), a\right) \\
&\leq d\left(z, g(z), x_{2n+1}\right) + d\left(z, x_{2n+1}, a\right) + \alpha \max\left\{d(x_{2n}, z, a),\right. \\
&\quad \left. d(x_{2n}, f(x_{2n}), a), d(z, g(z), a)\right\} + (1-\alpha)\left(p\, d(x_{2n}, g(z), a) +\right. \\
&\quad \left. q\, d(z, f(x_{2n}), a)\right) \\
&= d\left(z, g(z), x_{2n+1}\right) + d\left(z, x_{2n+1}, a\right) + \alpha \max\left\{d(x_{2n}, z, a),\right. \\
&\quad \left. d\left(x_{2n}, x_{2n+1}, a\right), d(z, g(z), a)\right\} + (1-\alpha)\left(p\, d(x_{2n}, g(z), a) +\right. \\
&\quad \left. q\, d(z, x_{2n+1}, a)\right)
\end{aligned}$$

Letting $n \to \infty$ we get $d\left(z, g(z), a\right) \leq \alpha\, d\left(z, g(z), a\right) + (1-\alpha)\ p\ d\left(z, g(z), a\right)$

$$= \left(\alpha + (1-\alpha)\, p\right)\ d\left(z, g(z), a\right)$$

$\Rightarrow \left(1-\alpha\right)\left(1-p\right) d\left(z, g(z), a\right) \leq 0$

$\Rightarrow d\left(z, g(z), a\right) = 0$ for every $a \in X$

$\Rightarrow z = g(z)$

$\Rightarrow z$ is a fixed point of g in X

Similarly, we prove that z is a fixed point of f in X. Hence z is a common fixed point of f and g in X.

1.3 Uniqueness of the Fixed Point

Let w be another common fixed point of f and g in X.

Then $f(w) = g(w) = w$.

Then $d(z,w,a)$ = $d\left(f(z), g(w), a\right)$

$$\leq \alpha \max\left\{d(z,w,a), d(z,f(z),a), d\left(w,g(w),a\right)\right\} + (1-\alpha)$$

$$\left(p\, d(z,g(w),a) + q\, d(w,f(z),a)\right)$$

$$= \alpha\, d(z,w,a) + (1-\alpha)\left(p+q\right) d(z,w,a)$$

$$= \left(\alpha + (1-\alpha)(p+q)\right) d(z,w,a)$$

$\Rightarrow\ d(w,z,a) \leq 0$

$\Rightarrow\ d\left(w,z,a\right) = 0$

$\Rightarrow\ w = z$

Hence z is the unique common fixed point of f and g in X.

1.3.1 Remark

The following corollary – 2.3 is a fixed point theorem that was proved by *Lal* and *Singh*. It can be easily proved by taking $\alpha = k_1 + k_2 + k_3$, $p = \dfrac{k_4}{1-\alpha}$ and $q = \dfrac{k_5}{1-\alpha}$ in our generalized fixed point theorem – 2.1.

1.3.2 Corollary [5]

Let f and g be two self-maps on a complete 2-metric space (X,d) such that

(a) $d(f(x), g(y), a) \leq k_1\, d(x, y, a) + k_2\, d(x, f(x), a) + k_3\, d(y, f(x), a) +$
$k_4\, d(x, g(y), a) + k_5\, d(y, g(y), a)$
for all x, y, a in X and each $k_i \geq 0$.

(b) $\sum_{i=1}^{5} k_i < 1$

(c) $|k_4 - k_5|\ (k_1 + k_2 + k_3) < 1 - \sum_{i=1}^{5} k_i$

Then f and g have a unique common fixed point in X.

1.3.3 Theorem

Suppose that $p \geq 1$ and $0 < h < 1$. Let $\{f_n : n = 0,1,2,...\}$ be a sequence of self-maps on a complete 2-metric space $\left(X,d\right)$ such that

$$d(f_0^{\,p}(x), f_n^{\,p}(y), a) \leq h \max.\left\{d(x,y,a), d(x,f_0^{\,p}(x),a), d(y,f_n^{\,p}(y),a),\right.$$
$$\left.\tfrac{1}{2}\left(d(x,f_n^{\,p}(y),a) + d(y,f_0^{\,p}(x),a)\right)\right\}$$

For every x, y, a in X. Then there exists a unique common fixed point of f_n in X.

Proof: Fix $x_0 \in X$. Define a sequence $\{x_n\}$ in X as follows.

$x_{2n-1} = f_0{}^p(x_{2n-2})$

$x_{2n} = f_n{}^p(x_{2n-1})$ where $n = 0,1,2,3...$.

First we prove that $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$ for $n = 0,1,2,...$

We have $d\left(x_{2n}, x_{2n+1}, x_{2n+2}\right) = d\left(x_{2n+1}, x_{2n+2}, x_{2n}\right)$

$$= d(f_0{}^p(x_{2n}), f_{n+1}{}^p(x_{2n+1}), x_{2n})$$

$$\le h \max.\{d(x_{2n}, x_{2n+1}, x_{2n}), d(x_{2n}, f_0{}^p(x_{2n}), x_{2n}),$$

$$d\left(x_{2n+1}, f_{n+1}{}^p(x_{2n+1}), x_{2n}\right), \tfrac{1}{2}\left[d(x_{2n}, f_{n+1}{}^p(x_{2n+1}), x_{2n})\right.$$

$$\left.+\ d(x_{2n+1}, f_0{}^p(x_{2n}), x_{2n})\right]\}$$

$$= h\, d\left(x_{2n+1}, f_{n+1}{}^p(x_{2n+1}), x_{2n}\right)$$

$$= h\, d\left(x_{2n+1}, x_{2n+2}, x_{2n}\right)$$

$$= h\, d\left(x_{2n}, x_{2n+1}, x_{2n+2}\right)$$

$$\Rightarrow d(x_{2n}, x_{2n+1}, x_{2n+2}) \le h\, d\left(x_{2n}, x_{2n+1}, x_{2n+2}\right)$$

$$\Rightarrow d\left(x_{2n}, x_{2n+1}, x_{2n+2}\right) = 0$$

Now $d\left(x_1, x_2, a\right) = d\left(f_0{}^p(x_0), f_1{}^P(x_1), a\right)$

$$\le h \max.\{d(x_0, x_1, a), d\left(x_0, f_0{}^p(x_0), a\right), d\left(x_1, f_1{}^p(x_1), a\right),$$

$$\tfrac{1}{2}\left[d\left(x_0, f_1^p(x_1), a\right) + d(x_1, f_0{}^p(x_0), a)\right]\}$$

$$= h \max\{d(x_0, x_1, a), d(x_1, x_2, a), \tfrac{1}{2}\, d(x_0, x_2, a)\}$$

$$\le h \max.\{d(x_0, x_1, a), d(x_1, x_2, a), \tfrac{1}{2}\left[d(x_1, x_2, a) + d(x_0, x_1, a)\right]\}$$

Put $M = \max\{d(x_0, x_1, a), d(x_1, x_2, a), \tfrac{1}{2}\left[d(x_1, x_2, a) + d(x_0, x_1, a)\right]\}$

If $M = d\left(x_1, x_2, a\right)$ then we get $d\left(x_1, x_2, a\right) \le h\, d\left(x_1, x_2, a\right)$ which is a contradiction.

If $M = \tfrac{1}{2}\left(d\left(x_0, x_1, a\right) + d\left(x_1, x_2, a\right)\right)$

Then $d\left(x_1, x_2, a\right) \le \tfrac{h}{2}\left(d\left(x_0, x_1, a\right) + d\left(x_1, x_2, a\right)\right)$

Now $d\left(x_0, x_1, a\right) \le \tfrac{h}{2}\left(d\left(x_0, x_1, a\right) + d\left(x_1, x_2, a\right)\right)$

$\Rightarrow d\left(x_0, x_1, a\right) + d\left(x_1, x_2, a\right) \le h\left(d\left(x_0, x_1, a\right) + d\left(x_1, x_2, a\right)\right)$ which is also a contradiction.

We must have $M = d\left(x_0, x_1, a\right)$

Then $d\left(x_1, x_2, a\right) \le h\, d\left(x_0, x_1, a\right)$

Similarly $d\left(x_2, x_3, a\right) \le h^2\, d\left(x_0, x_1, a\right)$

Continuing in this way we get $d(x_n, x_{n+1}, a) \le h^n d(x_0, x_1, a)$

We have $d\left(x_n, x_{n+m}, a\right) \le d\left(x_n, x_{n+m}, x_{n+1}\right) + d\left(x_n, x_{n+1}, a\right) + d\left(x_{n+1}, x_{n+m}, a\right)$

$$\le \sum_{k=1}^{n+m-2} d\left(x_k, x_{k+1}, x_{n+m}\right) + \sum_{k=n}^{n+m-1} d\left(x_k, x_{k+1}, a\right)$$

But $d\left(x_k, x_{k+1}, x_{n+m}\right) \le h^k d\left(x_0, x_1, x_{n+m}\right)$ and $d\left(x_k, x_{k+1}, a\right) \le h^k d\left(x_0, x_1, \mathrm{a}\right)$

$$\Rightarrow \sum_{k=1}^{n+m-2} d\left(x_k, x_{k+1}, x_{n+m}\right) \leq \sum_{k=1}^{n+m-2} h^k d\left(x_0, x_1, x_{n+m}\right)$$

$$< h^n \frac{1}{1-h} d\left(x_0, x_1, x_{n+m}\right) \to 0 \text{ as } n \to \infty.$$

Similarly $\sum_{k=n}^{n+m-1} d\left(x_k, x_{k+1}, a\right) \to 0$ as $n \to \infty$.

Hence $d\left(x_n, x_{n+m}, a\right) \to 0$ as $n \to \infty$.

$\Rightarrow \{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges some point z in X.

Now we prove that z is a unique common fixed point to each f_n, $n = 0,1,2,...$

Now $d\left(z, f_0^p(z), a\right) \leq d\left(z, f_0^p(z), x_{2n}\right) + d(z, x_{2n}, a) + d\left(x_{2n}, f_0^p(z), a\right)$

$$= d(z, x_{2n}, a) + d\left(z, f_0^p(z), x_{2n}\right) + d\left(f_n^p(x_{2n-1}), f_0^p(z), a\right)$$

$$\leq d(z, x_{2n}, a) + d\left(z, f_0^p(z), x_{2n}\right) + h \max \{d(z, x_{2n-1}, a),$$

$$d(z, f_0^p(z), a), d(x_{2n-1}, x_{2n}, a),$$

$$\tfrac{1}{2}\left[d(z, x_{2n}, a) + d(x_{2n-1}, f_0^p(z), a)\right]\}$$

When $n \to \infty$ and $x_n \to z$

$$d\left(z, f_0^p(z), a\right) \leq h \max \left\{d(z, f_0^p(z), a), \tfrac{1}{2} d(z, f_0^p(z), a)\right\}$$

$$\Rightarrow d\left(z, f_0^p(z), a\right) \leq h\, d\left(z, f_0^p(z), a\right)$$

$$\Rightarrow d\left(z, f_0^p(z), a\right) = 0 \ \forall\ a \in X$$

$$\Rightarrow f_0^p(z) = z$$

$\Rightarrow$ z is a fixed point of f_0^p.

Also we have $d\left(z, f_n^p(z), a\right) = d\left(f_0^p(z), f_n^p(z), a\right)$

$$\leq h \max \{d(z, z, a), d\left(z, f_0^p(z), a\right), d\left(z, f_n^p(z), a\right),$$

$$\tfrac{1}{2}\left(d\left(z, f_n^p(z), a\right) + d\left(z, f_0^p(z), a\right)\right)\}$$

$$= h \max \left\{d\left(z, f_n^p(z), a\right), \tfrac{1}{2} d\left(z, f_n^p(z), a\right)\right\}$$

$$= h\, d\left(z, f_n^p(z), a\right)$$

$$\therefore d\left(z, f_n^p(z), a\right) \leq h\, d\left(z, f_n^p(z), a\right)$$

$$\Rightarrow d\left(z, f_n^p(z), a\right) = 0$$

$$\Rightarrow f_n^p(z) = z$$

Hence z is a fixed point of f_n^p.

$\therefore z$ is a common fixed point of f_0^p and f_n^p $(n = 1,2,...)$. If possible $z \neq w$ be another fixed point.

Then $f_0^p(w) = f_n^p(w) = w$.

Now $d\left(z,w,a\right) = d\left(f_0^{\,p}(z), f_n^{\,p}(w), a\right)$

$$\leq h \max \left\{d\left(z,w,a\right), d\left(z, f_0^{\,p}(z), a\right), d\left(w, f_n^{\,p}(w), a\right), \tfrac{1}{2}\left(d\left(z, f_n^{\,p}(w), a\right) + d\left(w, f_0^{\,p}(z), a\right)\right)\right\}$$

$$= h \max \left\{d\left(z,w,a\right), 0, 0, \tfrac{1}{2}\left(d(z,w,a) + d(z,w,a)\right)\right\}$$

$$= h\, d\left(z,w,a\right)$$

$$\therefore d\left(z,w,a\right) \leq h\, d\left(z,w,a\right)$$

$\Rightarrow d\left(z,w,a\right) = 0$

$\Rightarrow z = w$

$\therefore$ z is a unique common fixed point of $f_0^{\,p}$ and $f_n^{\,p}$.

If we put $p = 1$, then z is a unique common fixed point of f_n in X, where $n = 0,1,2,3,...$

1.3.4 Corollary

Let $0 < h < 1$. Let f and g be two self-maps on a complete 2-metric space $\left(X, d\right)$ satisfying

$$d\left(f(x), g(y), a\right) \leq h \max \left\{d(x,y,a), d(x, f(x), a), d(y, g(y), a), \tfrac{1}{2}\left(d(x, g(y), a) + d(y, f(x), a)\right)\right\}$$

for every x, y, a in X. Then f and g have a unique common fixed point in X.

2. CONCLUSION

The fixed point theorems proved by Lal and Singh in [5] become corollaries to the generalized fixed point theorem that is established in this research article. Another fixed point theorem and a corollary are proved independently and these theorems can be further generalized to generalized metric spaces, $G-$metric spaces etc....

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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